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Stationary Increment on the Set Indexed Stochastic Process

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Abstract: *In this article, a new definition of stationary increment is presented. Here, stationary increment is a strengthened definition of classical stationary increment. Therefore, we are able to define and extend the following topics on set indexed framework: correlation, continuity (standard continuous, α-Holder continuous, mean square continuous and P-continuous), differentiability (standard derivative and mean square derivative), path independent variation, estimation, Brownian motion etc.*

Keywords*: Set indexed, continuity, differentiability, stationary increment and correlation.*

Introduction

Studying a stochastic process with stationary increment is a classical research topic in probability. As we know, a stochastic process $X=\{X_t\}$ in discrete or continuous time t such that the statistical characteristics of its increments of some fixed order, do not vary over time (that is, are invariant with respect to the time shifts $t \to t+\Delta t$). As in the case of stationary stochastic processes, one distinguishes two types of such processes, stochastic processes with stationary increments in the strict sense and in the wide sense. The correlation theory of stochastic processes with stationary increments in the wide sense was developed by Kolmogorov (see [Ko]). Theory of continuity, differentiability and other issues in the stochastic processes with stationary increment you can see in [Do], [It], [Ya], [Pi].

In this study, a new stationary increment is presented, which is defined on set indexed stochastic process. Set indexed processes are a natural generalization of planar processes where $\mathbf{A}$ is a collection of compact subsets of a fixed topological space (T,τ). The choice of the collection $\mathbf{A}$ is critical: it must be sufficiently rich in order to generate the Borel sets of T, but small enough to ensure the existence of a continuous Gaussian process defined on $\mathbf{A}$. This stationary increment is a strengthened definition of classical stationary increment: Let $X=\{X_A : A\in\mathbf{A}\}$ be a set-indexed stochastic process. X is said to have σ-stationary increment if $X_{A_1^\varepsilon}-X_{A_1} \overset{d}{=} \ldots \overset{d}{=} X_{A_n^\varepsilon}-X_{A_n}$ for all $\{A_i\}_{i=1}^n \in \mathbf{A}$, for all $\varepsilon>0$ and for all $A_i^\varepsilon \in D_{A_i}^\varepsilon$ (the notation $\overset{d}{=}$ mean identical distribution) when A^ε is the element in $D_A^\varepsilon=\{B\in\mathbf{A} : A\subseteq B, \sigma_{B\setminus A}=\varepsilon\}$ and σ is a positive and continuous measure in $\mathbf{A}$.

As we know, Random Walk, Brownian motion, Poisson processes, Renewal processes and other important processes are defined by stationary increment, then stationary increment is one of the most important topics in the stochastic process. With the definition of σ-stationary increment, we are able to define and extend the following topics on set indexed framework: correlation, continuity (standard continuous, α-Holder continuous, mean square continuous and P-continuous), differentiability (standard derivative and mean square derivative), path independent variation, estimation, Brownian motion etc.

Preliminaries

The set-indexed framework:

Let (T,τ) denote a non-void σ -compact connected topological space. In set indexed works (see [IvMe], [Sa]), processes and filtrations will be indexed by a nonempty class $\mathbf{A}$ of compact connected subsets of T is called an indexed collection if it satisfies the following:

a. $\varnothing \in \mathbf{A}$. In addition, there is an increasing sequence (B_n) of sets in $\mathbf{A}$ such that $T=\bigcup_{n=1}^{\infty} B_n^{\circ}$.

b. $\mathbf{A}$ is closed under arbitrary intersections and if $A,B\in\mathbf{A}$ are nonempty, then $A\bigcap B$ is nonempty. If (A_i) is an increasing sequence in $\mathbf{A}$ and if there exists n such that $A_i \subseteq B_n$ for every i, then $\overline{\bigcup_i A_i}\in\mathbf{A}$.

c. $\sigma(\mathbf{A})=\mathbf{B}$ where $\mathbf{B}$ is the collection of Borel sets of T.

d. There exist an increasing sequence of finite sub-classes $\mathbf{A}_n=\{A_1^n,...,A_{k_n}^n\}\subseteq\mathbf{A}$ closed under intersection with $\varnothing, B_n\in\mathbf{A}_n(\mathbf{u})$ ($\mathbf{A}_n(\mathbf{u})$ is the class of union of sets in $\mathbf{A}_n$), and a sequence of functions $g_n:\mathbf{A}\to\mathbf{A}_n(\mathbf{u})\cup T$ such that:

   i. g_n preserves arbitrary intersections and finite unions.

   ii. For each $A\in\mathbf{A},\ A\subseteq g_n(A)^{\circ}$ and $A=\cap_n g_n(A)$, $g_n(A)\subseteq g_m(A)$ if $n\geq m$

   iii. $g_n(A)\cap A'\in\mathbf{A}$ if $A,A'\in\mathbf{A}$ and $g_n(A)\cap A'\in\mathbf{A}_n$ if $A\in\mathbf{A}$ and $A'\in\mathbf{A}_n$.

   iv. $g_n(\varnothing)=\varnothing$ for all n.

(Note: $\overline{(\cdot)}$ and $(\cdot)^{\circ}$ denote respectively the closure and the interior of a set).

Examples of topological spaces T and indexed collections $\mathbf{A}$:

a. The classical example is $T=\Re_+^d$ and $\mathbf{A}=\mathbf{A}(\Re_+^d)=\{[0,x]:x\in\Re_+^d\}$.

b. The example (a) may be generalized as follows. Let $T=\Re_+^d$ and take $\mathbf{A}$ to be the class of compact *lower sets*, i.e. the class of compact subsets A of T satisfying $t\in A$ implies $[0,t]\subseteq\mathbf{A}$ (We denote the class of compact *lower sets* by $\mathbf{A}(Ls)$).

We define three extensions of $\mathbf{A}$: $\mathbf{A}(\mathbf{u})$ which consists of all finite unions in $\mathbf{A}$, $\mathbf{C}$ which consists of all set differences of the form $A\setminus B$ ($A\in\mathbf{A}, B\in\mathbf{A}(\mathbf{u})$) and $\mathbf{C}(\mathbf{u})$ which consists of all finite unions in $\mathbf{C}$. We note that $\mathbf{A}(\mathbf{u})$ is itself a lattice with the partial order induced by set inclusion.

Let (Ω,F,P) be any complete probability space. A set indexed filtration is a class $\{F_A:A\in\mathbf{A}\}$ of complete sub-σ-algebras of F which satisfies the following conditions:

a. $\forall A,B\in\mathbf{A}$, $F_A\subseteq F_B$, if $A\subseteq B$

b. Monotone outer-continuity: $F_{\bigcap A_i}=\bigcap F_{A_i}$ for any decreasing sequence (A_i) in $\mathbf{A}$.

For consistency in what follows, if $T\notin\mathbf{A}$ define $F_T=F$. In addition, let A^{ss} be any finite sub-semilattice of $\mathbf{A}$ closed under intersection. For $A\in A^{ss}$, define the left neighborhood of A in A^{ss} to be a set $C_A=A\setminus\bigcup_{B\in A^{ss},B\subset A}B$. We note that $\bigcup_{A\in A^{ss}}A=\bigcup_{A\in A^{ss}}C_A$ and that the latter union is disjoint. The sets in A^{ss} can always be numbered in the following way: $A_0=\varnothing'$, ($\varnothing'=\bigcap_{A\in\mathbf{A},A\neq\varnothing}A$, note that $\varnothing'\neq\varnothing$) and given $A_0,...,A_{i-1}$, choose A_i to be any set in A^{ss} such that $A\subset A_i$ implies that $A=A_j$, some $j=1,...,i-1$. Any such numbering $A^{ss}=\{A_0,...,A_k\}$ will be called "consistent with the strong past" (i.e., if C_i is the left-neighborhood of A_i in A^{ss}, then $C_i=\bigcup_{j=0}^{i}A_j\setminus\bigcup_{j=0}^{i-1}A_j$ and $C_i\bigcap A_j=\varnothing$, for all $j=0,...,i-1,\ i=1,...,k$).

Any $\mathbf{A}$ -indexed function which has a (finitely) additive extension to $\mathbf{C}$ will be called additive and is easily seen to be additive on $\mathbf{C(u)}$ as well. For stochastic processes, we do not necessarily require that each sample path be additive, but additivity will be imposed in an almost sure sense:

A set-indexed stochastic process $X = \{X_A : A \in \mathbf{A}\}$ is additive if it has an (almost sure) additive extension to $\mathbf{C}: X_\varnothing = 0$ and if $C, C_1, C_2 \in \mathbf{C}$ with $C = C_1 \bigcup C_2$ and $C_1 \bigcap C_2 = \varnothing$ then almost surely $X_C = X_{C_1} + X_{C_2}$. In particular, if $C \in \mathbf{C}$ and $C = A \setminus \bigcup_{i=1}^n A_i$, $A, A_1, ..., A_n \in \mathbf{A}$ then almost surely

$$X_C = X_A - \sum_{i=1}^{n} X_{A \cap A_i} + \sum_{i<j} X_{A \cap A_i \cap A_j} - ... + (-1)^n X_{A \cap \bigcap_{i=1}^n A_i} .$$

We shall always assume that our stochastic processes are additive. We note that a process with an (almost sure) additive extension to $\mathbf{C}$ also has an (almost sure) additive extension to $\mathbf{C(u)}$.

Let $X = \{X_A : A \in \mathbf{A}\}$ be an integrable additive set-indexed stochastic process and adapted with respect to filtration $F = \{F_A : A \in \mathbf{A}\}$. X is said to be a martingale if for any $A, B \in \mathbf{A}$ such that $A \subseteq B$, we have $E[X_B \mid F_A] = X_A$. For a study of the different kinds of martingales see [MeNu], [Za], [Kh].

Definition 1. Let $[a,b] \subset \Re_+$. A strict flow (shortly, flow) is defined to be a continuous increasing function $f : [a,b] \to \mathbf{A(u)}$, i.e. such that

a. $\forall s,t \in [a,b]$; $s < t \Rightarrow f(s) \subset f(t)$
b. $\forall s,t \in [a,b)$; $f(s) = \bigcap_{v>s} f(v)$
c. $\forall s,t \in (a,b)$; $f(s) = \overline{\bigcup_{u<s} f(u)}$.

The notion of flow was introduced in [CaWa] and used by several authors [Da], [He].

Given a set indexed stochastic process X and the flow $f : [a,b] \to \mathbf{A(u)}$, we define a process Y indexed by $[a,b]$ as follows: $Y_s = X_{f(s)} = X_s^f$ for all $s \in [a,b]$.

Definition 2.

a. A positive measure σ on $(T, \mathbf{B})$ is called strictly monotone on $\mathbf{A}$ if: $\sigma_{\varnothing'} = 0$ and $\sigma_A < \sigma_B$ for all $A \subset B$, $A, B \in \mathbf{A}$.
b. Let σ be a positive, strictly monotone on $\mathbf{A}$ and continuous measure in $\mathbf{A}$. If $A \in \mathbf{A}$ and $\varepsilon \ge 0$ then define $D_A^\varepsilon = \{B \in \mathbf{A} : A \subseteq B, \sigma_{B \setminus A} = \varepsilon\}$. Denote by A^ε the element in D_A^ε and assume that $D_A^\varepsilon \neq \varnothing$. The collection of these measures is denoted by $M(\mathbf{A})$. (Note: If $\varepsilon = 0$ then $A^\varepsilon = A^0 = A$)

Hereafter, we assume that the space T has a positive and continuous measure $\sigma \in M(\mathbf{A})$ in $\mathbf{A}$ such that for all $A \in \mathbf{A}$ there exists a A^ε, $\sigma_{A^\varepsilon \setminus A} = \varepsilon$ for all $\varepsilon \ge 0$.

The classical examples are:

a. $T = \Re_+^d$ and $\mathbf{A} = \mathbf{A}(\Re_+^d) = \{[0,x] : x \in \Re_+^d\}$ when σ is Lebesgue measure or Radon measure.
b. $T = \Re_+^d$ and $\mathbf{A} = \mathbf{A}(Ls)$ when σ is Lebesgue measure.

Definition 3. Let $X = \{X_A : A \in \mathbf{A}\}$ be a set-indexed stochastic process.

a. X is said to have σ-stationary increment if $X_{A_1^\varepsilon} - X_{A_1} \overset{d}{=} ... \overset{d}{=} X_{A_n^\varepsilon} - X_{A_n}$ for all $\{A_i\}_{i=1}^n \in \mathbf{A}$, for all $\varepsilon > 0$ and for all $A_i^\varepsilon \in D_{A_i}^\varepsilon$ (the notation $\overset{d}{=}$ mean identical distribution).

b. X is said to have independent increment if $X_{C_1},\ldots,X_{C_n}$ are independent random variables whenever $C_1,\ldots,C_n$ are disjoint sets in $\mathbf{C}$.

Path independent variation:

A continuous two-parameter martingale is said to have path-independent variation if the quadratic variation along all increasing and continuous paths from the origin and with the same end point has the same value. This notion was introduced by Wong and Zakai [WoZa]. It was shown in [CaWa] that continuous strong martingales have path-independent variation, and the reciprocal implication is not true in general. Here we extend it to the set-indexed framework.

Let $X=\{X_t : t\geq 0\}$ be a square integrable martingale. It is know that we can associate with X a unique predictable process denoted $\langle X\rangle$ such that $X^2-\langle X\rangle$ is a martingale. Few is known in the set-indexed case ([IvMe07]). However, the concept of flow and σ-stationary increment permits to study such processes.

Definition 4. A set-indexed process $X=\{X_A : A\in\mathbf{A}\}$ is said L^2-monotone outer-continuous if X_A is:

a. Square-integrable for all $A\in\mathbf{A}$ (In other words, $E(X_A^2)<\infty$ for all $A\in\mathbf{A}$).
b. For any decreasing sequence $\{A_n\}_{n=1}^{\infty}\in\mathbf{A}$, $\lim_{n\to\infty}E[|X_{A_n}-X_{\bigcap_{m=1}^{\infty}A_m}|^2]=0$.

Theorem 1. (Path independent variation) Let $\sigma\in M(\mathbf{A})$ and $X=\{X_A : A\in\mathbf{A}\}$ be a set-indexed L^2-monotone outer-continuous, independent increment and σ-stationary increment then X has path independent variation.
(X is said to have path independent variation if for any flows $f_1,f_2:[0,\infty)\to\mathbf{A}(\mathbf{u})$ and with $f_1(0)=f_2(0)$ and $f_1(b)=f_2(b)$, $0<b<\infty$, then $\langle X^{f_1}\rangle(b)=\langle X^{f_2}\rangle(b)$. This definition was introduced by Cairoli and Walsh in the plane, see [CaWa]).

Proof.
Let $f:[0,\infty)\to\mathbf{A}(\mathbf{u})$ a flow. The process X is an outer-continuous then X^f is right-continuous. X is independent increment then X is a strong martingale (Particular, X is a martingale). Therefore, X^f is a martingale. Moreover, X is σ-stationary increment then X^f is time-change right-continuous martingale, independent increment and stationary increment. (In other words, for all a flow $f:[0,\infty)\to\mathbf{A}(\mathbf{u})$ there exists an increasing function $\theta:[0,\infty)\to[0,\infty)$ such that $X^{f\circ\theta}$ is right-continuous martingale, independent increment and stationary increment). More details about θ (time-change) you can see in [Yo], [MeYo].
Now, for any flows $f_1,f_2:[0,\infty)\to\mathbf{A}(\mathbf{u})$, $X^{f_1\circ\theta_1}$ and $X^{f_2\circ\theta_2}$ are is right-continuous martingale, independent increment and stationary increment then (We recall that, let $X=\{X_t : t\geq 0\}$ is a right-continuous martingale and $EX_t^2<\infty$ for all $t\geq 0$. If X is independent increment and stationary increment then $\langle X\rangle_t=tEX_1^2$ for all $t\geq 0$), then $\langle X^{f_1\circ\theta_1}\rangle,\langle X^{f_2\circ\theta_2}\rangle$ is deterministic and in particular $\langle X^{f_1\circ\theta_1}\rangle(b)=\langle X^{f_2\circ\theta_2}\rangle(b)$ there $f_1(\theta_1(b))=f_2(\theta_2(b))$.

Proposition: If $\sigma_{f(b)}=k$ there exists a unique $s\geq 0$ such that $f(\theta(s))=f(b)$ and $s=k$ for all a flow $f:[0,\infty)\to\mathbf{A}(\mathbf{u})$.
Proof.
It is clear that $f(\theta(k))=k$ from definition θ ($\theta(t)=\alpha^{-1}(t)$ when $\alpha:[0,\infty)\to[0,\infty)$ $\alpha(t)=\sigma_{f(t)}$) then $f(\theta(k))=f(b)$, else, there exists $r\neq k$ when $f(\theta(r))=f(b)$ and $f(\theta(r))\neq f(\theta(k))$ because of strict continuous (flow). Without loss of generality, we may assume that $f(\theta(r))\subset f(\theta(k))$ when $\sigma_{f(\theta(r))}=\sigma_{f(\theta(k))}$, which is a contradiction to $\sigma\in M(\mathbf{A})$.

Let $f_1, f_2 : [0,\infty) \to \mathbf{A}(\mathbf{u})$ be two flows when $f_1(0) = f_2(0)$ and $f_1(b) = f_2(b)$ then $\sigma_{f_1(b)} = \sigma_{f_2(b)}$. Based on Proposition, we imply that $f_1(b) = f_1(\theta_1(k))$, $f_2(b) = f_2(\theta_2(k))$ then $\langle X^{f_1} \rangle(b) = \langle X^{f_1 \circ \theta_1} \rangle(k) = \langle X^{f_2 \circ \theta_2} \rangle(k) = \langle X^{f_2} \rangle(b)$. □

Continuity and differentiability

Definition 5. Let $X = \{X_A : A \in \mathbf{A}\}$ be a set-indexed stochastic process.

a. X is said to be continuous at $A \in \mathbf{A}$ if $\lim\limits_{\varepsilon \to 0^+} X_{A^\varepsilon} = X_A$ for all $A^\varepsilon \in D_A^\varepsilon$ (the limit is mean in the sense of almost surely convergence).

b. X is said to be α-Holder continuous at $A \in \mathbf{A}$ if there exists $M > 0, \delta > 0$ such that $\left| X_{A^\varepsilon} - X_A \right| \le M \varepsilon^\alpha$ for all $0 < \varepsilon < \delta$, for all $A^\varepsilon \in D_A^\varepsilon$ and $0 < \alpha \le 1$.

c. X is said to be mean square continuous (shortly, MS continuous) at $A \in \mathbf{A}$ if $\lim\limits_{\varepsilon \to 0^+} E\left[(X_{A^\varepsilon} - X_A)^2 \right] = 0$.

d. X is said to be P-continuous at $A \in \mathbf{A}$ if $\lim\limits_{\varepsilon \to 0^+} P\left[\left| X_{A^\varepsilon} - X_A \right| > \beta \right] = 0$ for all $\beta > 0$.

e. X is said to be differentiable at $A \in \mathbf{A}$ if there exists a random variable Y such that $\lim\limits_{\varepsilon \to 0^+} \frac{X_{A^\varepsilon} - X_A}{\varepsilon} - Y = 0$ for all $A^\varepsilon \in D_A^\varepsilon$ and denote $Y = X'_A$ (the limit is mean in the sense of almost surely convergence).

f. X is said to be mean square differentiable (shortly, MS differentiable) at $A \in \mathbf{A}$ if there exists a random variable Y such that $\lim\limits_{\varepsilon \to 0^+} E\left[\left(\frac{X_{A^\varepsilon} - X_A}{\varepsilon} - Y \right)^2 \right] = 0$ for all $A^\varepsilon \in D_A^\varepsilon$ and denote $Y = X_A^{diff}$.

Note: It is clear,

1. If X is a α-Holder continuous at $A \in \mathbf{A}$ then X is a continuous at $A \in \mathbf{A}$.
2. If X is a differentiable at $A \in \mathbf{A}$ then X is a MS differentiable at $A \in \mathbf{A}$.

Let X be a set-indexed square-integrable. The autocorrelation of a process X is by definition the mean of the product $X_A X_B$. This function, will be denote by $R_X(A,B)$ or $R(A,B)$. Thus, $R_X(A,B) = R(A,B) = E[X_A X_B]$.

If also X have σ-stationary increment then the autocorrelation of a process X will be denote by $R_X(A, A^\varepsilon)$ or $R(\varepsilon)$. Thus, $R_X(A, A^\varepsilon) = R(\varepsilon) = E[X_A X_{A^\varepsilon}]$.

Theorem 2. Let $X = \{X_A : A \in \mathbf{A}\}$ be a square-integrable, σ-stationary increment set-indexed stochastic process.

a. R is continuous at $0 \Leftrightarrow X$ is MS continuous at $A \in \mathbf{A}$.

b. If R is continuous at 0 then its mean is continuous ($\lim\limits_{\varepsilon \to 0^+} EX_{A^\varepsilon} = EX_A$).

c. If R is continuous at 0 and X continuous at $A \in \mathbf{A}$ then $\lim\limits_{\varepsilon \to 0^+} EX_{A^\varepsilon} = E\left[\lim\limits_{\varepsilon \to 0^+} X_{A^\varepsilon} \right]$.

d. If X is MS continuous at $A \in \mathbf{A}$ then X is P-continuous at $A \in \mathbf{A}$.

e. The second generalized derivative of R exists at all $\varepsilon \ge 0$ ($\frac{d^2 R}{d\varepsilon^2}$ is exists) $\Leftrightarrow X$ is MS differentiable at all $A \in \mathbf{A}$

f. If X is MS differentiable at $A \in \mathbf{A}$ then X is MS continuous at $A \in \mathbf{A}$.

g. If X is differentiable at $A \in \mathbf{A}$ then X is continuous at $A \in \mathbf{A}$.

Proof.

a. It is clear that $E\left[(X_{A^\varepsilon}-X_A)^2\right]=R(A^\varepsilon,A^\varepsilon)-2R(A^\varepsilon,A)+R(A,A)=2(R(0)-R(\varepsilon))$

Then $\lim_{\varepsilon\to0^+} E\left[(X_{A^\varepsilon}-X_A)^2\right]=0 \iff \lim_{\varepsilon\to0^+} R(0)-R(\varepsilon)=0 \iff \lim_{\varepsilon\to0^+} R(\varepsilon)=R(0)$.

b. As we know, $\left[E(X_{A^\varepsilon}-X_A)\right]^2 \le E\left[(X_{A^\varepsilon}-X_A)^2\right]$. If R is continuous at 0 then based on (b), X is MS continuous at $A\in\mathbf{A}$. Hence,

$$0\le \lim_{\varepsilon\to0^+}\left[E(X_{A^\varepsilon}-X_A)\right]^2 \le \lim_{\varepsilon\to0^+} E\left[(X_{A^\varepsilon}-X_A)^2\right]=0.$$

c. According to (b), $\lim_{\varepsilon\to0^+} EX_{A^\varepsilon}=E[X_A]=E\left[\lim_{\varepsilon\to0^+} X_{A^\varepsilon}\right]$.

d. Based on a Chebyshev inequality, $P\left[\left|X_{A^\varepsilon}-X_A\right|>\beta\right]\le \frac{E[(X_{A^\varepsilon}-X_A)^2]}{\beta}$ for all $\beta>0$. Then

$$0\le \lim_{\varepsilon\to0^+} P\left[\left|X_{A^\varepsilon}-X_A\right|>\beta\right]\le \tfrac{1}{\beta}\lim_{\varepsilon\to0^+} E\left[\left(X_{A^\varepsilon}-X_A\right)^2\right]=\tfrac{1}{\beta}\cdot 0=0.$$

e. ($\Rightarrow$) It suffices to show that (Cauchy criterion) $\lim_{\varepsilon,\delta\to0^+} E\left[\left(\frac{X_{A^\varepsilon}-X_A}{\varepsilon}-\frac{X_{A^\delta}-X_A}{\delta}\right)^2\right]=0$.

We use this criterion because, Definition 5(f), it does not involve the unknown X_A^{diff}. Obvious,

$$E\left[\left(X_{A^\varepsilon}-X_A\right)\left(X_{A^\delta}-X_A\right)\right]=R(A^\varepsilon,A^\delta)-R(A^\varepsilon,A)-R(A,A^\delta)+R(A,A).$$

Thus, $\lim_{\varepsilon,\delta\to0^+} E\left[\left(\frac{X_{A^\varepsilon}-X_A}{\varepsilon}-\frac{X_{A^\delta}-X_A}{\delta}\right)^2\right]=$

$$=2\lim_{\varepsilon\to0^+}\frac{R(A^\varepsilon,A^\varepsilon)-R(A^\varepsilon,A)-R(A,A^\varepsilon)+R(A,A)}{\varepsilon^2}-2\lim_{\varepsilon,\delta\to0^+}\frac{R(A^\varepsilon,A^\delta)-R(A^\varepsilon,A)-R(A,A^\delta)+R(A,A)}{\varepsilon\delta}$$

We assumed that $\frac{d^2R}{d\varepsilon^2}$ exists for all $\varepsilon\ge0$, thus $\lim_{\varepsilon,\delta\to0^+} E\left[\left(\frac{X_{A^\varepsilon}-X_A}{\varepsilon}-\frac{X_{A^\delta}-X_A}{\delta}\right)^2\right]=0$.

($\Leftarrow$) Obvious.

f. $\lim_{\varepsilon\to0^+} E\left[(X_{A^\varepsilon}-X_A)^2\right]=\lim_{\varepsilon\to0^+} E\left[\frac{(X_{A^\varepsilon}-X_A)^2}{\varepsilon^2}\cdot\varepsilon^2\right]=\lim_{\varepsilon\to0^+}\varepsilon^2\lim_{\varepsilon\to0^+} E\left[\left(\frac{X_{A^\varepsilon}-X_A}{\varepsilon}\right)^2\right]=0$.

g. $\lim_{\varepsilon\to0^+}(X_{A^\varepsilon}-X_A)=\lim_{\varepsilon\to0^+}\frac{X_{A^\varepsilon}-X_A}{\varepsilon}\cdot\varepsilon=\lim_{\varepsilon\to0^+}\varepsilon\lim_{\varepsilon\to0^+}\frac{X_{A^\varepsilon}-X_A}{\varepsilon}=0\cdot X'_A=0$. □

The cross-correlation of two processes $X=\{X_A : A\in\mathbf{A}\}$ and $Y=\{Y_A : A\in\mathbf{A}\}$ is the function $R_{X,Y}(A,B)=E[X_AY_B]$.

Lemma 1. Let $X=\{X_A : A\in\mathbf{A}\}$ be a square-integrable, σ-stationary increment set-indexed stochastic process.

a. If X is differentiable at $B\in\mathbf{A}$ then $R_{X,X'}(A,B)=E[X_AX'_B]=\frac{dR_X(A,B)}{dB}$ when $\frac{dR_X(A,B)}{dB}=\lim_{\varepsilon\to0^+}\frac{R_X(A,B^\varepsilon)-R_X(A,B)}{\varepsilon}$.

b. If X is differentiable at $A\in\mathbf{A}$ then $R_{X,X'}(A,A)=R_{X',X}(A,A)=R'(0)$

c. If X is differentiable at all $A\in\mathbf{A}$ and R' exists at all $0\le\varepsilon$ then

$$R_{X',X'}(A,A)=-R''(0)\ ,\ R_{X',X'}(A^\varepsilon,A)=-R''(\varepsilon).$$

Proof.

a. $R_{X,X'}(A,B) = E[X_A X'_B] = E[X_A \lim_{\varepsilon\to 0^+} \frac{X_{B^\varepsilon}-X_B}{\varepsilon}] = \lim_{\varepsilon\to 0^+} E[X_A \frac{X_{B^\varepsilon}-X_B}{\varepsilon}] =$

$$= \lim_{\varepsilon\to 0^+} E[X_A \frac{X_{B^\varepsilon}-X_B}{\varepsilon}] = \lim_{\varepsilon\to 0^+} \frac{R_X(A,B^\varepsilon)-R_X(A,B)}{\varepsilon} = \frac{dR_X(A,B)}{dB}$$

b. Based on (a), $R_{X,X'}(A,A) = \lim_{\varepsilon\to 0^+} \frac{R_X(A,A^\varepsilon)-R_X(A,A)}{\varepsilon} = \lim_{\varepsilon\to 0^+} \frac{R(\varepsilon)-R(0)}{\varepsilon} = R'(0)$.

c. Let $A\in\mathbf{A}$ and $A^\varepsilon \in D_A^\varepsilon$. There exists a flow $f:[0,\infty)\to\mathbf{A}(\mathbf{u})$ and $0<s<t$ such that $f(s)=A,\ f(t)=A^\varepsilon$ (see the proof in [Yo], [MeYo]) . We define a process Y indexed by $[0,\infty)$ as follows: $Y_t = X_{f(t)} = X_t^f$ for all $t\in[0,\infty)$. We assumed that X is differentiable for all $A\in\mathbf{A}$, then Y is differentiable for all $t\in[0,\infty)$. Thus, $\frac{\partial^2 R_Y(t,s)}{\partial t\partial s} = \frac{\partial^2 R_Y(t-s)}{\partial t\partial s} = \frac{\partial}{\partial t}(\frac{dR_Y(\varepsilon)}{\partial\varepsilon}\cdot\frac{\partial(t-s)}{\partial s}) = -\frac{d^2R_Y(\varepsilon)}{\partial\varepsilon^2}\cdot\frac{\partial\varepsilon}{\partial t} = -R_Y''(\varepsilon) = -R_{X^f}''(\varepsilon) = -R''(\varepsilon)$.
□

Note: In the same way, we can prove that if $X_A^{(n)}$ exists ($X_A^{(n)}$-the n-th derivative of X at $A\in\mathbf{A}$) for all $A\in\mathbf{A}$, $A^\varepsilon\in D_A^\varepsilon$ and $R^{(n+k)}$ exists at $0\le\varepsilon$ then for all $1\le k\le n$

$$R_{X^{(n)},X^{(k)}}(A^\varepsilon,A) = (-1)^{n+k+1}R^{(n+k)}(\varepsilon)\ ,\ R_{X^{(n)},X^{(k)}}(A,A) = (-1)^{n+k+1}R^{(n+k)}(0)$$

Theorem 3. (Estimation) Let $X=\{X_A : A\in\mathbf{A}\}$ be a square-integrable, σ-stationary increment set-indexed stochastic process. If R has derivatives of all orders, and if $X_A^{(n)}$ exists ($X_A^{(n)}$-the n-th derivative of X at $A\in\mathbf{A}$) for all n and $R(\varepsilon)=\sum_{n=0}^{\infty} R^{(n)}(0)\frac{\varepsilon^n}{n!}$ for all $0<\varepsilon$, then the estimation of X_{A^ε} is $\sum_{n=0}^{\infty} X_A^{(n)}\frac{\varepsilon^n}{n!}$ for all $A\in\mathbf{A}$ and $A^\varepsilon\in D_A^\varepsilon$. (In other words, $E\left[\left(X_{A^\varepsilon}-\sum_{n=0}^{\infty} X_A^{(n)}\frac{\varepsilon^n}{n!}\right)^2\right]=0$).

(Note: Let X,Y be a random variables with finite variance. We say that estimation of Y is X if $E[(Y-X)^2]$ is minimal (see [Pa])).

Proof.

Enough to prove that $E\left[\left(X_{A^\varepsilon}-Y\right)^2\right]=0$ when $Y=\sum_{n=0}^{\infty} X_A^{(n)}\frac{\varepsilon^n}{n!}$. We assumed that $R(\varepsilon)=\sum_{n=0}^{\infty} R^{(n)}(0)\frac{\varepsilon^n}{n!}$ then $R^{(m)}(\varepsilon)=\sum_{n=0}^{\infty} R^{(n+m)}(0)\frac{\varepsilon^n}{n!}$. Therefore, based on Lemma 1

$$E\left[\left(X_{A^\varepsilon}-Y\right)X_A^{(m)}\right] = (-1)R^{(m)}(\varepsilon) - \sum_{n=0}^{\infty}(-1)^m R^{(n+m)}(0)\frac{\varepsilon^n}{n!} = 0$$

Y is a linear combination of $X_A^{(m)}$ then $E\left[\left(X_{A^\varepsilon}-Y\right)Y\right]=0$. Clearly, $R(0)=R(\varepsilon-\varepsilon)=\sum_{n=0}^{\infty} R^{(n)}(\varepsilon)\frac{(-\varepsilon)^n}{n!}$ then $E\left[\left(X_{A^\varepsilon}-Y\right)X_{A^\varepsilon}\right] = R(0)-\sum_{n=0}^{\infty}(-1)^n R^{(n)}(\varepsilon)\frac{\varepsilon^n}{n!}=0$.

Hence, $E\left[\left(X_{A^\varepsilon}-Y\right)^2\right] = E\left[\left(X_{A^\varepsilon}-Y\right)X_{A^\varepsilon}\right] - E\left[\left(X_{A^\varepsilon}-Y\right)Y\right] = 0$. □

Definition 6. Let $\sigma\in M(\mathbf{A})$. We say that the $\mathbf{A}$-indexed process X is a Brownian motion with variance σ if X can be extended to a finitely additive process on $\mathbf{C}(\mathbf{u})$ and if for disjoint sets $C_1,\dots,C_n\in\mathbf{C}$, $X_{C_1},\dots,X_{C_n}$ are independent mean-zero Gaussian random variables with variances $\sigma_{C_1},\dots,\sigma_{C_n}$, respectively. (For any $\sigma\in M(\mathbf{A})$, there exists a set-indexed Brownian motion with variance σ [IvMe]).

Lemma 2. Let $\sigma \in M(\mathbf{A})$ and $X = \{X_A : A \in \mathbf{A}\}$ be a set indexed Brownian motion. If $\{A_i\}_{i=1}^{\infty}$ (or $\{A_i\}_{i=1}^{k}$) be an increasing sequence in $\mathbf{A}(\mathbf{u})$ then there exists a flow $f:[0,\infty) \to \mathbf{A}(\mathbf{u})$, $f(0) = \emptyset'$ and $f(i) = A_i$ for all $1 \le i$, such that X^f is a time-change Brownian motion. (The process X^f is called a time-change Brownian motion if there exists $\theta:[0,\infty) \to [0,\infty)$ such that $X^{f \circ \theta}$ is a Brownian motion).
You can see the proof in [Yo].

Theorem 4. (α-Holder continuity and non-differentiability) Let $\sigma \in M(\mathbf{A})$ and $X = \{X_A : A \in \mathbf{A}\}$ be a set indexed Brownian motion with variance σ.

a. If $\alpha < \frac{1}{2}$ then X is a α-Holder continuous path almost everywhere.
b. X is not differentiable at $A \in \mathbf{A}$, for almost all ω. (In other words, set indexed Brownian motion is nowhere differentiable almost surely).

Proof. Let $A \in \mathbf{A}$ and $A^{\varepsilon} \in D_A^{\varepsilon}$. According to Lemma 2, there exists a flow $f:[0,\infty) \to \mathbf{A}(\mathbf{u})$ and there exists $0 \le t \le t^{\varepsilon}$ such that X^f is a time-change Brownian motion and $A^{\varepsilon} = f(t^{\varepsilon}), A = f(t)$. Thus, there exists $\theta:[0,\infty) \to [0,\infty)$ and $0 \le \beta \le \beta^{\varepsilon}$ such that $X^{f\circ\theta}$ is a Brownian motion and $A^{\varepsilon} = f(t^{\varepsilon}) = f(\theta(\beta^{\varepsilon})),\ A = f(t) = f(\theta(\beta))$

a. $X^{f\circ\theta}$ is a Brownian motion (We recall that, if $W = \{W_t : t \ge 0\}$ is a classical Brownian motion and $0 < \alpha < \frac{1}{2}$, then W is a α-Holder continuous path almost everywhere), then there exists a $M > 0, \delta > 0$ such that $\left|X_{\beta^{\varepsilon}}^{f\circ\theta} - X_{\beta}^{f\circ\theta}\right| \le M\varepsilon^{\alpha}$ for all $0 < \varepsilon < \delta$ and $0 < \alpha < \frac{1}{2}$. But $\left|X_{\beta^{\varepsilon}}^{f\circ\theta} - X_{\beta}^{f\circ\theta}\right| = \left|X_{A^{\varepsilon}} - X_A\right|$ then there exists a $M > 0, \delta > 0$ such that $\left|X_{A^{\varepsilon}} - X_A\right| \le M\varepsilon^{\alpha}$ for all $0 < \varepsilon < \delta$, for all $A^{\varepsilon} \in D_A^{\varepsilon}$ and $0 < \alpha < \frac{1}{2}$.
b. $X^{f\circ\theta}$ is a Brownian motion, then $\lim\limits_{\varepsilon \to 0^+} \frac{X_{A^{\varepsilon}} - X_A}{\varepsilon} = \lim\limits_{\varepsilon \to 0^+} \frac{X_{t^{\varepsilon}}^{f} - X_t^{f}}{\varepsilon} = \lim\limits_{\varepsilon \to 0^+} \frac{X_{\beta^{\varepsilon}}^{f\circ\theta} - X_{\beta}^{f\circ\theta}}{\varepsilon}$ is not differentiable. □

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