g$#_\psi$ - closed sets in topological spaces

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Abstract: In this paper, we introduce a new class of generalized closed sets called $g^\#_\psi$ - closed sets which contains the class of $\psi$ - closed sets and contained in the class of $\psi g$ - closed sets in topological spaces. We obtain the relations between $g^\#_\psi$ - closed sets and other existing closed sets. Also we study the properties $g^\#_\psi$ - closed sets.

Keywords: $\psi$ - closed sets, $\psi g$ - closed sets and $g^\#_\psi$ - closed sets

1. Introduction


2. Preliminaries

Throughout this paper $(X, \tau)$ represents non - empty topological space on which no separation axioms are assumed unless otherwise mentioned. For a subset $A$ of a space $(X, \tau)$, $cl(A)$ and $int(A)$ denote the closure of $A$ and the interior of $A$ respectively.

Definition 2.1: A subset $A$ of a topological space $(X, \tau)$ is called

1) Regular - open set [9] if $A = int(cl(A))$
2) Semi - open set [3] if $A \subseteq cl(int(A))$
3) $\alpha$ - open set if $A \subseteq cl(int(cl(A)))$

The complements of the above mentioned sets are called regular - closed, semi - closed and $\alpha$ - closed respectively.

The intersection of all regular - closed (resp., semi - closed and $\alpha$ - closed) subsets of $(X, \tau)$ containing $A$ is called the regular - closure (resp. semi - closure and $\alpha$ - closure) of $A$ and is denoted by $rcl(A)$ (resp. $scl(A)$ and $\alpha cl(A)$).

Definition 2.2: A subset $A$ of a topological space $(X, \tau)$ is called

1) generalized closed set (briefly g - closed)
2) $g\#_\psi$ - closed set if $\psi cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\psi g$ - open in $(X, \tau)$.
3) Generalized $\alpha$ - closed set (briefly $g\#_\alpha$ - closed) [7] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\alpha$ - open in $(X, \tau)$.
4) $g^\#_\psi$ - closed set [10] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $g$ - open in $(X, \tau)$.
5) $\psi g$ - closed set [12] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is semi-open in $(X, \tau)$.
6) $\psi g$ - closed set [11] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\psi g$ - open in $(X, \tau)$.
7) $\alpha g$ - closed set [15] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\alpha g$ - open in $(X, \tau)$.
8) $\psi$ - closed set [11] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\psi g$ - open in $(X, \tau)$.
9) $\psi g^\#_\psi$ - closed set [13] if $\psi cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $g$ - open in $(X, \tau)$.
10) $\psi g^\#_\psi$ - closed set [11] if $\psi cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $g$ - open in $(X, \tau)$.
11) $\psi g^\#_\psi$ - closed set [12] if $\psi cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $g$ - open in $(X, \tau)$.
12) $\psi g^\#_\psi$ - closed set [13] if $\psi cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $g$ - open in $(X, \tau)$.

The complements of the above mentioned sets are called their respective open sets.

3. $g^\#_\psi$ - closed sets

Definition 3.1 A subset $A$ of a topological space $(X, \tau)$ is called $g^\#_\psi$ - closed, if $\psi cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\psi$ - open in $(X, \tau)$.
Example 3.2 Let $X = \{a, b, c\}$ with topology $\tau = \{X, \phi, \{a\}\}$. Then the subsets $X, \phi, \{b\}, \{c\}$ and $\{b, c\}$ are $g^*\psi$ - closed.

Proposition 3.3 Every closed set in $(X, \tau)$ is $g^*\psi$ - closed in $(X, \tau)$ but not conversely.

Proof: Let A be a closed set in $(X, \tau)$ and U be any $\psi$ - open set containing A in $(X, \tau)$. Since A is closed, $cl(A) = A$. For every subset A of X, $\psi cl(A) \subseteq cl(A) = A \subseteq U$ and so we have $\psi cl(A) \subseteq U$. Hence A is $g^*\psi$ - closed.

Example 3.4 Let $X = \{a, b, c\}$ with topology $\tau = \{X, \phi, \{a\}\}$. Then the subset $\{b\}$ is $g^*\psi$ - closed but not closed.

Proposition 3.5 Every regular - closed set in $(X, \tau)$ is $g^*\psi$ - closed in $(X, \tau)$ but not conversely.

Proof: The proof follows from the result that any regular - closed set is closed and by Proposition 3.3.

Example 3.6 Let $X = \{a, b, c\}$ with topology $\tau = \{X, \phi, \{a\}\}$. Then the subset $\{b\}$ is $g^*\psi$ - closed but not regular - closed set.

Proposition 3.7 Every $\alpha$ - closed set in $(X, \tau)$ is $g^*\psi$ - closed in $(X, \tau)$ but not conversely.

Proof: Let A be an $\alpha$ - closed set and U be any $\psi$ - open set containing A. Since A is $\alpha$ - closed, $acl(A) = A$. For every subset A of X, $\psi acl(A) \subseteq acl(A) = A \subseteq U$ and so we have $\psi acl(A) \subseteq U$. Hence A is $g^*\psi$ - closed.

Example 3.8 Let $X = \{a, b, c\}$ with topology $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. Then the subset $\{b\}$ is $g^*\psi$ - closed but not $\alpha$ - closed.

Proposition 3.9 Every semi - closed set in $(X, \tau)$ is $g^*\psi$ - closed in $(X, \tau)$ but not conversely.

Proof: Let A be a semi - closed set and U be any $\psi$ - open set containing A. Since A is semi - closed, $scl(A) = A$. For every subset A of X, $\psi scl(A) \subseteq scl(A) = A \subseteq U$ and so we have $\psi scl(A) \subseteq U$. Hence A is $g^*\psi$ - closed.

Example 3.10 Let $X = \{a, b, c\}$ with topology $\tau = \{X, \phi, \{a\}, \{b\}\}$. Then the subset $\{a, c\}$ is $g^*\psi$ - closed but not semi - closed.

Proposition 3.11 Every $\psi$ - closed set in $(X, \tau)$ is $g^*\psi$ - closed in $(X, \tau)$ but not conversely.

Proof: Let A be a $\psi$ - closed set and U be any $\psi$ - open set containing A. Since A is $\psi$ - closed, $\psi cl(A) = A \subseteq U$. Hence A is $g^*\psi$ - closed.

Example 3.12 Let $X = \{a, b, c\}$ with topology $\tau = \{X, \phi, \{a\}, \{b\}\}$. Then the subset $\{b\}$ is $g^*\psi$ - closed but not $\psi$ - closed.

Proposition 3.13 Every $\psi^*\psi$ - closed set in $(X, \tau)$ is $g^*\psi$ - closed in $(X, \tau)$ but not conversely.

Proof: Let A be a $\psi^*\psi$ - closed set and U be any $\psi$ - open set containing A. Since every $\psi$ - open set is $\psi$ - open and A is $\psi^*\psi$ - closed, $\psi cl(A) \subseteq U$. Hence A is $g^*\psi$ - closed.

Example 3.14 Let $X = \{a, b, c\}$ with topology $\tau = \{X, \phi, \{a\}, \{b\}\}$. Then the subset $\{a, b\}$ is $g^*\psi$ - closed but not $\psi$ - closed.

Proposition 3.15 Every $g^*\psi$ - closed set in $(X, \tau)$ is $\psi$ - closed in $(X, \tau)$ but not conversely.

Proof: Let A be a $g^*\psi$ - closed set and U be any $\psi$ - open set containing A. Since every open set is $\psi$ - open and A is $g^*\psi$ - closed, $\psi cl(A) \subseteq U$. Hence A is $\psi$ - closed.

Example 3.16 The following examples show that $g^*\psi$ - closedness is independent from $\psi$ - closedness.

Example 3.17 Let $X = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}\}$. In this topology the subset $\{c\}$ is $g^*\psi$ - closed but not $g^*\psi$ - closed.

Example 3.18 Let $X = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{b\}\}$. In this topology the subset $\{a, b\}$ is $g^*\psi$ - closed but not $g^*\psi$ - closed.

Remark 3.19 The following examples show that $g^*\psi$ - closedness is independent from $\alpha$ - closedness.

Example 3.20 Let $X = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{b\}\}$. In this topology the subset $\{a\}$ is $g^*\psi$ - closed, but not $\alpha$ - closed.

Example 3.21 Let $X = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{b\}\}$. In this topology the subset $\{a, c\}$ is $\alpha$ - closed but not $g^*\psi$ - closed.

Remark 3.22 The following examples show that $g^*\psi$ - closedness is independent from $\psi$ - closedness.

Example 3.23 Let $X = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{b\}\}$. In this topology the subset $\{a, c\}$ is $g^*\psi$ - closed, but not $g^*\psi$ - closed.
Example 3.24 Let $X = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{b, c\}\}$. In this topology the subset $\{b\}$ is $g^\psi$-closed but not $g^\psi$-closed.

Remark 3.25 The following examples show that $g^\psi$-closedness is independent from $\psi\hat{g}$-closedness and $g^\psi$-closedness.

Example 3.26 Let $X = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}\}$. In this topology the subset $\{a, b\}$ is $g^\psi$-closed and $g^\psi$-closed but not $g^\psi$-closed.

Example 3.27 Let $X = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{b, c\}\}$. In this topology the subset $\{a, b\}$ is $g^\psi$-closed but not $\psi\hat{g}$-closed and not $g^\psi$-closed.

Remark 3.28 The following examples show that $g^\psi$-closedness is independent from $g^\#\psi$-closedness.

Example 3.29 Let $X = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. In this topology the subset $\{b\}$ is $g^\#\psi$-closed but not $g^\#\psi$-closed.

Example 3.30 Let $X = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{b\}\}$. In this topology the subset $\{a, c\}$ is $g^\#\psi$-closed but not $g^\#\psi$-closed.

Remark 3.31 Union of two $g^\psi$-closed sets need not be $g^\psi$-closed sets as seen from the following example.

Example 3.32 Let $X = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. Then the subsets $X, \phi, \{a\}, \{b\}, \{c\}, \{a, c\}$ and $\{b, c\}$ are $g^\psi$-closed but $\{a\}$ and $\{b, c\}$ are $g^\psi$-closed but not $g^\psi$-closed.

Remark 3.33 The following diagrams show the relationship between $g^\psi$-closed sets with already existing closed sets.

\[\text{Diagram}\]

where $A \rightarrow B$ represents $A$ implies $B$ and $A \leftrightarrow B$ represents $A$ and $B$ are independent.

Definition 3.39 A subset $A$ of a topological space $(X, \tau)$ is said to be $g^\psi$-open if its complement $A^c$ is $g^\psi$-closed.

The class of all $g^\psi$-open sets in $(X, \tau)$ is denoted by $g^\#\psi O(X, \tau)$.

Proposition 3.40 Every $g^\#\psi$-open set is $\psi\hat{g}$-open.

4. Properties of $g^\#\psi$-closed sets and $g^\#\psi$-open sets

Theorem 4.1 If $A$ is a $g^\psi$-closed subset of $(X, \tau)$ and $A \subseteq B \subseteq \psi\cl(A)$. Then $B$ is also a $g^\psi$-closed set in $(X, \tau)$.

Proof: Let $U$ be any $\psi$-open set in $(X, \tau)$ such that $B \subseteq U$. Then $A \subseteq U$, Since $A$ is $g^\psi$-closed, $\psi\cl(A) \subseteq U$. Also since $B \subseteq \psi\cl(A)$, $\psi\cl(B) \subseteq \psi\cl(\psi\cl(A)) = \psi\cl(A)$. Hence $\psi\cl(B) \subseteq U$. Therefore $B$ is also a $g^\psi$-closed set in $(X, \tau)$.

Theorem 4.2 Let $A$ be $g^\psi$-closed set in $(X, \tau)$, then $\psi\cl(A) - A$ contains no non-empty closed set.

Proof: Suppose that $A$ is $g^\psi$-closed in $(X, \tau)$.
Let F be a closed subset of ψcl(A) – A. Then F is open and hence ψ-open such that A ⊆ F. Since A is gψ-open, ψcl(A) ⊆ F. Thus

F ⊆ (ψcl(A))ψ. Since every closed set is ψ-closed, F is ψ-closed. Hence F ⊆ ψcl(A) – A. Therefore F ⊆ ψcl(A) ∩ (ψcl(A))ψ = ϕ. Hence F = ϕ.

**Theorem 4.3** A set A is gψ-open if and only if ψcl(A) – A contains no non-empty ψ-closed set.

**Proof:** (Necessity) Let A be gψ-open subset of X. Let F be a ψ-closed set contained in ψcl(A) – A. Since F is ψ-open with A ⊆ F and A is gψ-open, F contains no non-empty ψ-closed set of ψcl(A) – A, which is a contradiction. Therefore ψcl(A) ⊆ G and hence A is gψ-open.

**Sufficiency:** Let ψcl(A) – A contains no non-empty ψ-closed set. Let A ⊆ G and G be ψ-open. If ψcl(A) is not a subset of G then ψcl(A) ∩ G is a non-empty ψ-closed subset of ψcl(A) – A, which is a contradiction. Therefore ψcl(A) ⊆ G and hence A is gψ-open.

**Proposition 4.4** If a set A is ψ-open and gψ-open, then A is ψ-open.

**Proof:** Since A is ψ-open, A is gψ-open, ψcl(A) ⊆ A. Hence A is ψ-open.

**Theorem 4.5** If a set A is gψ-open and ψ-open, then A is ψ-open.

**Proof:** Since A is gψ-open and ψ-open, A is ψ-open (by Proposition 4.4). Since F is ψ-open in (X, ψ), F is ψ-open in (X, ψ).

**Theorem 4.6** For each x ∈ X either {x} is ψ-open or X - {x} is a gψ-open set in (X, ψ).

**Proof:** Let x ∈ X and suppose that {x} is not ψ-open. Then X - {x} is not ψ-open in X. Hence X is the only ψ-open set containing X - {x}. That is (X - {x}) ⊆ X. Therefore ψcl(X - {x}) ⊆ X which implies that X - {x} is gψ-open closed set in (X, ψ).

**Theorem 4.7** Let A be gψ-open closed set in (X, ψ). Then A is ψ-open if and only if ψcl(A) – A is closed.

**Proof:** (Necessity) Let A be an any ψ-open set relative to Y.
5. REFERENCES


