Imperial Journal of Interdisciplinary Research (IJIR)
Vol-3, Issue-4, 2017
ISSN: 2454-1362, http://www.onlinejournal.in

$g^{\#}\psi$ - closed sets in topological spaces

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Abstract : *In this paper, we introduce a new class of generalized closed sets called $g^{\#}\psi$ - closed sets which contains the class of ψ – closed sets and contained in the class of ψg - closed sets in topological spaces. We obtain the relations between $g^{\#}\psi$ - closed sets and other existing closed sets. Also we study the properties $g^{\#}\psi$ - closed sets.*
Keywords : *ψ - closed sets, ψg - closed sets and $g^{\#}\psi$ - closed sets*

1. Introduction

Levine [4] introduced the concept of generalized closed sets in topological spaces. Veerakumar [11] introduced and studied ψ- closed sets in topological spaces. Ramya and Parvathi [8] introduced a new concept of generalized closed sets called ψg - closed sets in topological spaces. In this paper we introduce a new class of sets namely $g^{\#}\psi$ - closed sets and study their basic properties.

2. Preliminaries

Throughout this paper (X, τ) represents non - empty topological space on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ), cl(A) and int(A) denote the closure of A and the interior of A respectively.

Definition 2.1: A subset A of a topological space (X, τ) is called

(1)Regular - open set [9] if A = int(cl(A))
(2)Semi - open set [3] if A ⊆ cl(int(A))
(3)α - open set if [7] if A ⊆ int(cl(int(A)))

The complements of the above mentioned sets are called regular - closed, semi - closed and α - closed respectively.

The intersection of all regular - closed (resp., semi-closed and α - closed) subsets of (X,τ) containing A is called the regular - closure (resp. semi – closure and α - closure) of A and is denoted by rcl(A) (resp. scl(A) and αcl(A)).

Definition 2.2:

A subset A of topological space of (x,τ) is called

1) generalized closed set (briefly g - closed) [6] if cl(A) ⊆ U whenever A ⊆ U and U is open in (X,τ).
2) Semi - generalized closed set (briefly sg - closed) [3] if scl(A) ⊆ U whenever A ⊆ U and U is semi- open in (X, τ).
3) Generalized α - closed set(briefly gα - closed) [7] if αcl(A) ⊆ U whenever A ⊆ U and U is α-open in (X,τ).
4) α - generalized closed set (briefly αg - closed)[8] if αcl(A) ⊆ U whenever A ⊆ U and U is open in (X,τ).
5) $\hat{g}$ - closed set [12] if cl(A)⊆U whenever A⊆U and U is semi-open in (X, τ).
6) g^{*}-closed set [10] if cl(A) ⊆U whenever A⊆U and U is g-open in (X, τ).
7) $^{*}g$- closed set [14] if cl(A) ⊆U whenever A⊆U and U is $\hat{g}$-open in (X, τ).
8) ψ- closed set [11] if scl(A) ⊆U whenever A⊆U and U is sg - open in (X, τ).
9) $g^{*}\psi$ - closed set [13] if ψcl(A) ⊆ U whenever A⊆U and U is g-open in (X, τ).
10) ψg - closed set [8] if ψcl(A)⊆U whenever A⊆U and U is open in (X, τ).
11) $\psi\hat{g}$ - closed set [8] if ψcl(A)⊆U whenever A⊆U and U is $\hat{g}$-open in (X, τ).
12) $\psi^{*}g^{*}$ - closed set [1] if ψcl(A)⊆U whenever A⊆U and U is ψg -open in (X, τ).

The complements of the above mentioned sets are called their respective open sets

3. $g^{\#}\psi$ - closed sets

Definition 3.1 A subset A of a topological space (X, τ) is called $g^{\#}\psi$ – closed, if ψcl(A) ⊆ U whenever A ⊆ U and U is ψ - open in (X, τ).

Example 3.2 Let X = {a, b, c} with topology τ ={X, ϕ, {a}}. Then the subsets X,ϕ, {b}, {c} and {b, c} are $g^{\#}\psi$ - closed.

Proposition 3.3 Every closed set in (X, τ) is $g^{\#}\psi$ - closed in (X, τ) but not conversely.

Proof: Let A be a closed set in (X, τ) and U be any ψ - open set containing A in (X, τ) . Since A is closed, cl(A) = A. For every subset A of X ψcl(A) ⊆ cl(A) = A ⊆ U and so we have ψcl(A) ⊆ U. Hence A is $g^{\#}\psi$ - closed.

Example 3.4 Let X = {a, b, c} with topology τ ={X, ϕ, {a}}. Then the subset {b} is $g^{\#}\psi$ - closed but not closed.

Proposition 3.5 Every regular - closed set in (X, τ) is $g^{\#}\psi$ - closed in (X, τ) but not conversely.

Proof: The proof follows from the result that every regular - closed set is closed and by **Proposition 3.3**.

Example 3.6 Let X = {a, b, c} with topology τ ={X, ϕ, {a}}. Then the subset {b} is $g^{\#}\psi$ - closed but not regular - closed set.

Proposition 3.7 Every α - closed set in (X, τ) is $g^{\#}\psi$ - closed in (X, τ) but not conversely.

Proof: Let A be an α - closed set and U be any ψ - open set containing A. Since A is α - closed αcl(A) = A. For every subset A of X ψcl(A) ⊆ αcl(A) = A ⊆ U and so we have ψcl(A) ⊆ U. Hence A is $g^{\#}\psi$ - closed.

Example 3.8 Let X = {a, b, c} with topology τ = {X, ϕ, {a}, {b},{a, b}}. Then the subset {b} is $g^{\#}\psi$ - closed but not α - closed.

Proposition 3.9 Every semi - closed set in (X, τ) is $g^{\#}\psi$ - closed in (X, τ) but not conversely.

Proof: Let A be a semi - closed set and U be any ψ - open set containing A. Since A is semi - closed scl(A) = A For every subset A of X ψcl(A) ⊆ scl(A) = A ⊆ U and so we have ψcl(A) ⊆ U. Hence A is $g^{\#}\psi$ - closed.

Example 3.10 Let X = {a, b, c} with topology τ = {X, ϕ, {a, b}}. Then the subset {a, c} is $g^{\#}\psi$ - closed but not semi - closed.

Proposition 3.11 Every ψ - closed set in (X, τ) is $g^{\#}\psi$ - closed in (X, τ) but not conversely.

Proof: Let A be a ψ - closed set and U be any ψ - open set containing A. Since A is ψ - closed, ψcl(A) = A ⊆ U. Hence A is $g^{\#}\psi$ - closed.

Example 3.12 Let X = {a, b, c} with topology τ = {X, ϕ, {a}, {b, c}}. Then the subset {b} is $g^{\#}\psi$- closed but not ψ - closed.

Proposition 3.13 Every $\psi^{*}g^{*}$ - closed set in (X, τ) is $g^{\#}\psi$ - closed in (X, τ) but not conversely.

Proof: Let A be a $\psi^{*}g^{*}$- closed set and U be any ψ - open set containing A. Since every ψ - open set is ψg – open and A is $\psi^{*}g^{*}$- closed, ψcl(A) ⊆ U. Hence A is $g^{\#}\psi$ – closed.

Example 3.14 Let X = {a, b, c} with topology τ = {X, ϕ, {a}, {b, c}}. Then the subset {a, b} is $g^{\#}\psi$ - closed but not $\psi^{*}g^{*}$- closed.

Proposition 3.14 Every $g^{\#}\psi$ - closed set in (X, τ) is ψg - closed in (X, τ) but not conversely.

Proof: Let A be a $g^{\#}\psi$ - closed set and U be any open set containing A. Since every open set is ψ – open and A is $g^{\#}\psi$ – closed, ψcl(A) ⊆ U. Hence A is ψg – closed.

Example 3.15 Let X = {a, b, c} with topology τ = {X, ϕ, {a}}. Then the subset {a,b} is ψg - closed but not $g^{\#}\psi$- closed.

Remark 3.16 The following examples show that $g^{\#}\psi$ - closedness is independent from g^{*} - closedness.

Example 3.17 Let X = {a, b, c} with τ = {X, ϕ, {a}}. In this topology the subset {c} is $g^{\#}\psi$ - closed but not g^{*} - closed.

Example3.18 Let X = {a, b, c} with τ = {X, ϕ, {a}, {a, b}}. In this topology the subset {a,c} is g^{*}- closed but not $g^{\#}\psi$ - closed.

Remark 3.19 The following examples show that $g^{\#}\psi$ - closedness is independent from αg - closedness.

Example 3.20 Let X = {a, b, c} with τ = {X, ϕ, {a}, {b}, {a, b}}. In this topology the subset {a} is $g^{\#}\psi$ - closed, but not αg - closed.

Example 3.21 Let X = {a, b, c} with τ = {X, ϕ, {a},{a, b}}. In this topology the subset {a,c} is αg - closed but not $g^{\#}\psi$ - closed.

Remark 3.22 The following examples show that $g^{\#}\psi$ - closedness is independent from $g^{*}\psi$ closedness.

Example 3.23 Let X = {a, b, c} with τ = {X,ϕ, {a}, {a, b}}.In this topology the subset {a,c} is $g^{*}\psi$ - closed, but not $g^{\#}\psi$ - closed.

Example 3.24 Let X = {a, b, c} with τ = {X, ϕ, {a}, {b,c}}.In this topology the subset {b}is $g^{\#}\psi$ - closed but not $g^{*}\psi$ - closed.

Remark 3.25 The following examples show that $g^{\#}\psi$ - closedness is independent from $\psi\hat{g}$ - closedness and *g-closedness.

Example 3.26 Let X = {a, b, c} with τ = {X, ϕ, {a}}.In this topology the subset {a, b} is $\psi\hat{g}$ - closed and *g - closed but not $g^{\#}\psi$ - closed.

Example 3.27 Let X = {a, b, c} with τ = {X, ϕ, {a}, {b, c}}.In this topology the subset {a, b} is $g^{\#}\psi$ - closed but not $\psi\hat{g}$ – closed and not *g - closed.

Remark 3.28 The following examples show that $g^{\#}\psi$ - closedness is independent from g -closedness.

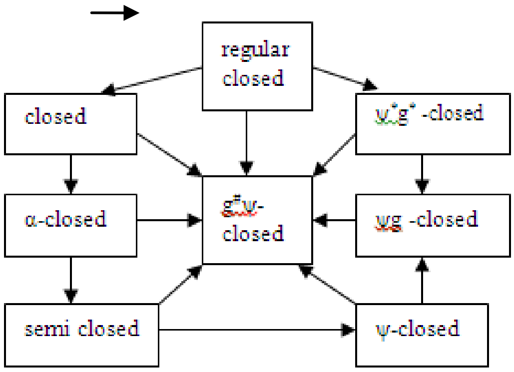
Example3.29 Let X = {a, b, c} with τ = {X, ϕ, {a}, {b}, {a, b}}.In this topology the subset {b} is $g^{\#}\psi$ - closed but not g - closed.

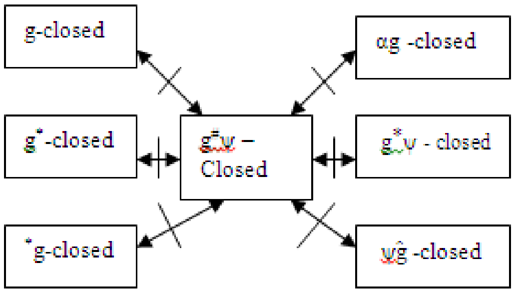
Example3.30 In X = {a, b, c} with τ = {X, ϕ, {a}}. In this topology the subset {a, c} is g - closed but not $g^{\#}\psi$ - closed.

Remark 3.31 Union of two $g^{\#}\psi$ - closed sets need not be $g^{\#}\psi$ - closed sets as seen from the following example.

Example3.32 Let X = {a, b, c} with τ = {X, ϕ, {a},{b},{a, b}}.Then the subsets X, ϕ, {a},{b},{c}, {a, c} and {b, c} are $g^{\#}\psi$ - closed but {a} U {b}={a, b} is not $g^{\#}\psi$ - closed.

Remark 3.33 The following diagrams show the relationship between $g^{\#}\psi$ – closed sets with already existing closed sets.





where A → B represents A implies B and
A ←|→ B represents A and B are independent.

Definition 3.39 A subset A of a topological space (X, τ) is said to be $g^{\#}\psi$ - open if its complement A^{c} is $g^{\#}\psi$ – closed.

The class of all $g^{\#}\psi$ - open sets in (X,τ) is denoted by $g^{\#}\psi$ O(X, τ).

Proposition 3.40 Every open (respectively Regular open, α - open, Semi open, ψ - open and $\psi^{*}g^{*}$ - open) set is $g^{\#}\psi$ - open.

Proposition 3.41 Every $g^{\#}\psi$ - open set is ψg - open.

4. Properties of $g^{\#}\psi$ -closed sets and $g^{\#}\psi$ -open sets

Theorem 4.1 If A is a $g^{\#}\psi$- closed subset of (X, τ) and A ⊆ B ⊆ ψcl(A). Then B is also a $g^{\#}\psi$ - closed set in (X, τ).

Proof: Let U be any ψ- open set in (X, τ) such that B⊆U. Then A ⊆ U, Since A is $g^{\#}\psi$ - closed, ψcl(A) ⊆ U. Also since B ⊆ ψcl(A), ψcl(B) ⊆ ψcl(ψcl(A)) = ψcl(A). Hence ψcl(B) ⊆ U. Therefore B is also a $g^{\#}\psi$- closed set in (X, τ).

Theorem 4.2 Let A be $g^{\#}\psi$ - closed set in (X, τ), then ψcl(A) – A contains no non - empty closed set.

Proof: Suppose that A is $g^{\#}\psi$ - closed in (X, τ).

Let F be a closed subset of ψcl(A) – A, Then F^c is open and hence ψ – open such that A ⊆ F^c. Since A is $g^\#\psi$- closed, ψcl(A) ⊆ F^c. Thus F ⊆ $(\psi cl(A))^c$. Since every closed set is ψ - closed, F is ψ - closed. Hence F ⊆ ψcl(A) –A. Therefore F ⊆ ψcl(A) ⋂ $(\psi cl(A))^c$ = ϕ. Hence F = ϕ.

Theorem 4.3 A set A is $g^\#\psi$ - closed in (X, τ) if and only if ψcl(A) – A contains no non - empty ψ - closed set.

Proof: **(Necessity)** Let A be $g^\#\psi$ - closed subset of X. Let F be a ψ - closed set contained in ψcl(A) – A. Since F^c is ψ - open with A ⊆ F^c and A is $g^\#\psi$ – closed set in X, ψcl(A) ⊆ F^c, Then F ⊆ $(\psi cl(A))^c$. Also F ⊆ ψcl(A) – A .Therefore F ⊆ $(\psi cl(A))^c$ ⋂ ψ cl(A) = ϕ. Hence F = ϕ.

Sufficiency: Let ψcl(A) – A contains no non - empty ψ - closed set. Let A ⊆ G and G be ψ -open. If ψcl(A) is not a subset of G the ψcl(A) ⋂ G^c is a non - empty ψ - closed subset of ψcl(A) – A, which is a contradiction. Therefore ψcl(A) ⊆ G and hence A is $g^\#\psi$ - closed.

Proposition 4.4 If a set A is ψ - open and $g^\#\psi$ – closed in (X, τ). Then A is a ψ - closed set of (X, τ).

Proof: Since A is ψ - open, $g^\#\psi$ - closed, ψcl(A) ⊆ A. Hence A is ψ- closed.

Theorem 4.5 If a set A is $g^\#\psi$ - closed and ψ – open and F is ψ- closed in (X, τ), then A⋂F is ψ- closed.

Proof: Since A is $g^\#\psi$ - closed and ψ – open, A is ψ - closed (by **Proposition 4.4**). Since F is ψ – closed in X, A ⋂ F is ψ – closed in (X, τ).

Theorem 4.6 For each x ∈ X either {x} is ψ - closed or X - {x} is a $g^\#\psi$ - closed set in (X, τ).

Proof: Let x ∈ X and suppose that {x} is not ψ - closed in X. Then X - {x} is not ψ - open in X. Hence X is the only ψ - open set containing X-{x}. That is (X-{x}) ⊆ X. Therefore ψcl(X - {x}) ⊆ X which implies that X -{x} is $g^\#\psi$ - closed set in (X, τ).

Theorem 4.7 Let A be $g^\#\psi$- closed set in (X, τ). Then A is ψ - closed if and only if ψcl(A) – A is closed.

Proof: (Necessity) Let A be an any ψ – closed subset of X. Then ψ cl(A) = A and so ψcl(A) – A = ϕ, which is closed.

Sufficiency: Let ψcl(A) – A be a closed set. Since A is $g^\#\psi$ - closed by **theorem 4.2**. ψcl(A) – A contains no non – empty closed set which implies ψcl(A) – A = ϕ. That is ψ cl(A) = A.
Hence A is ψ - closed.

Theorem 4.8 Let A be any $g^\#\psi$ - closed set of (X, τ). Then A is ψ - closed if and only if ψcl(A) – A is ψ - closed.

Proof: (Necessity) Let A be any ψ – closed subset of X. Then ψ cl(A) = A and so ψcl(A) – A=ϕ,Which is ψ - closed in (X, τ).

Sufficiency: Let ψcl(A) – A be a ψ- closed sets, Since A is $g^\#\psi$- closed by **theorem 4.3** ψcl(A) – A contains no non - empty ψ - closed set which implies ψcl(A) – A = ϕ. That is ψ cl(A) = A. Hence A is ψ - closed.

Theorem 4.9 Let A⊆B⊆ X and suppose that A is ψ - closed set in X then A is $g^\#\psi$ - closed set relative to Y.

Proof: Let A be a $g^\#\psi$- closed set in X, Let A ⊆ Y ⋂ U, where U is ψ- open set in X. Since A is $g^\#\psi$- closed, ψ cl(A) ⊆U. That is Y ⋂ ψ cl(A) ⊆ Y⋂U, where Y ⋂ ψ cl(A) is ψ closure of A in Y. Hence ψcl_Y (A) ⊆ Y ⋂ U. Thus A is $g^\#\psi$- closed set relative to Y.

Definition 4.10 The intersection of all ψ-open subsets of (X,τ) containing A is called ψ- kernal of A and is denoted by ψ-ker(A)
i.e ψ-ker(A)=∩{U / U is ψ-open in (X,τ) and A ⊆U}

Theorem 4.11 A subset A of (X,τ) is $g^\#\psi$ - closed in (X,τ) if and only if ψ cl(A) ⊆ψ-ker(A).

Proof: **(Necessity)** Suppose that A is $g^\#\psi$ -closed set in (X,τ). Let x∈ ψ cl(A). If x∉ψ-ker(A), then there exists a ψ- open set U in (X,τ) such that A⊆U and x∉U. Since U is ψ - open set containing A and A is $g^\#\psi$ –closed, we have ψcl(A) ⊆U, which is a contradiction to x∈ ψcl(A) and x∉U.

Sufficiency: Suppose that ψ cl(A) ⊆ ψ-ker(A). If U is any ψ -open set containing A, then ψcl(A) ⊆ ψ-ker(A) so we have ψcl(A) ⊆U. Hence A is $g^\#\psi$ - closed.

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