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Methods of finding square roots of 2×2 Matrices

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Abstract: *In this paper we have tried to find square roots of some 2×2 matrices of various types. It is found that every matrix has no square root. Some matrices have two square roots, while some matrices have four square roots. We have used general method of finding square roots of any 2×2 matrix. It has been verified that the method is found to suitable for finding square roots of 2×2 non singular matrices.*

1. Introduction:-

First we shall define the square root of matrix.
Definition:- If B is any non singular matrix of order $n\ x\ n$ then a matrix A of same order is called **square root** of matrix B $if\ A^2 = B$.

If $A=\begin{pmatrix} a & c \\ b & d \end{pmatrix}$,

Can we write $\sqrt{A} = \begin{pmatrix} \sqrt{a} & \sqrt{c} \\ \sqrt{b} & \sqrt{d} \end{pmatrix}$. The answer is No.

$A^2 = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a^2+bc & ac+cd \\ ab+bd & d^2+bc \end{pmatrix}$

Let $B = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$

Then $A^2 = B$ gives the relations
$a^2+bc = x, ac+cd = y$ ------ (1),
$ab+bd = z,\quad d^2+bc = w$ ----- (2)

Does every matrix has Square roots? No. Only non singular matrix has Square root.

Consider the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, whether it's inverse exists?

From equations (1) and (2), we get
$$a^2+bc = 0,\quad c(a+d) = 1,$$
$$b(a+d) = 0,\quad d^2+bc = 0\ .$$

As $c(a+d) = 1$, this implies c=1, $as\ a+d \neq 0$.
$b(a+d) = 0$, gives $b=0$, $d^2+bc = 0$, gives $d = 0$, $a^2+bc = 0$, gives $a = 0$.
Thus $a = 0, d = 0$, implies $a+d = 0$, which is a contradiction as $a+d \neq 0$.Hence there does not exists

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inverse of matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

1.Square root of Scalar Matrix:-

Let B= $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$,

Then $A^2 = B$ gives the relations
$a^2+bc = x,\quad c(a+d) = 0$,----(4)
$b(a+d) = 0,\quad d^2+bc = y$ ---- (5) The equations $b(a+d) = 0,\ c(a+d) = 0$ give $b = 0,\quad c = 0,\ \ a+d \neq 0$.
Hence $a^2 = x$, $d^2 = y$
i.e. $a = \pm\sqrt{x},\quad b = \pm\sqrt{y}$

Hence Square roots of Scalar Matrix $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$ are

$\begin{pmatrix} \sqrt{x} & 0 \\ 0 & \sqrt{y} \end{pmatrix}$, $\begin{pmatrix} -\sqrt{x} & 0 \\ 0 & -\sqrt{y} \end{pmatrix}$, $\begin{pmatrix} \sqrt{x} & 0 \\ 0 & -\sqrt{y} \end{pmatrix}$, $\begin{pmatrix} -\sqrt{x} & 0 \\ 0 & \sqrt{y} \end{pmatrix}$

For example the square roots of $\begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}$ are $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}, \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}$

2. Square root of matrix of the form $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

By usual equations, we get
$a^2+bc = 1,\quad c(a+d) = 1,\ b(a+d) = 0,\quad d^2+bc = 1.$
Solving these equations we obtain $a = \pm 1$, $b = 0,\ c = \pm\frac{1}{2},\ \ d = \pm 1$

Hence the Square roots of the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ are $\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} -1 & -\frac{1}{2} \\ 0 & -1 \end{pmatrix}$

3. Square root of matrix of the form $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$

By usual equations, we get
$$a^2+bc = x,\quad c(a+d) = y$$
$$b(a+d) = 0,\quad d^2+bc = z.$$
Solving these equations we obtain,
$a = \pm\sqrt{x},\quad b = 0\ ,\ c = \pm\frac{y}{(\sqrt{x}+\sqrt{z}))},\quad d = \pm\sqrt{z}.$

Hence the Square roots of the matrix of the form $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$ are $\begin{pmatrix} \pm\sqrt{x} & \pm\frac{y}{(\sqrt{x}+\sqrt{z}))} \\ 0 & \pm\sqrt{z} \end{pmatrix}$.

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Hence square roots of matrix $\begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix}$

are $\begin{pmatrix} \pm 1 & \pm 2/3 \\ 0 & \pm 2 \end{pmatrix}$.

i.e $\begin{pmatrix} 1 & 2/3 \\ 0 & 2 \end{pmatrix}$, $\begin{pmatrix} -1 & -2/3 \\ 0 & -2 \end{pmatrix}$

Square roots of matrix $\begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}$

are $\begin{pmatrix} \pm\sqrt{x} & \pm 1/2\sqrt{x} \\ 0 & \pm\sqrt{x} \end{pmatrix}$.

4. Square roots of general 2×2 matrix

Square roots of 2×2 matrices of the form $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ can be found using the relation

$\sqrt{A} = a_0 I + a_1 A$ --------- (1)

Where a_0, a_1 are to calculated from the relations

$\sqrt{\lambda_i} = a_0 I + a_1 \lambda_i$ ------------ (2)

λ_i are the Eigen values of matrix $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

Example (1): - Find the square root of Matrix $A = \begin{pmatrix} 1 & 2 \\ 5 & 4 \end{pmatrix}$.

Solution: - Eigen values of matrix A are given by $\begin{vmatrix} 1-\lambda & 2 \\ 5 & 4-\lambda \end{vmatrix} = 0$,

i.e. $\lambda^2 - 5\lambda - 6 = 0$

The Eigen values of matrix A are $\lambda_1 = 6$, $\lambda_2 = -1$.

Hence from the equations (2), we get the relations $\sqrt{6} = a_0 + 6a_1$ and $i = a_0 - a_1$

Solving these relations we obtain the values,

$a_0 = \frac{\sqrt{6}+6i}{7}$, $a_1 = \frac{\sqrt{6}-i}{7}$

Putting the values of a_0, a_1 in (1), we get

$\sqrt{A} = \frac{1}{7}\begin{pmatrix} \sqrt{6}+6i & 0 \\ 0 & \sqrt{6}+6i \end{pmatrix} + \frac{1}{7}\begin{pmatrix} \sqrt{6}-i & 2\sqrt{6}-2i \\ 5\sqrt{6}-5i & 4\sqrt{6}-4i \end{pmatrix}$.

$= \frac{1}{7}\begin{pmatrix} 2\sqrt{6}+5i & 2\sqrt{6}-2i \\ 5\sqrt{6}-5i & 5\sqrt{6}+2i \end{pmatrix}$.

It can be easily verified that $\sqrt{A} \times \sqrt{A} = \begin{pmatrix} 1 & 2 \\ 5 & 4 \end{pmatrix}$.

Example (2):- Find the square root of Matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 4 \end{pmatrix}$.

Solution: - Eigen values of matrix A are given by $\begin{vmatrix} 1-\lambda & 1 \\ 0 & 4-\lambda \end{vmatrix} = 0$,

i.e. $\lambda^2 - 5\lambda + 4 = 0$.

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The Eigen values of matrix A are $\lambda_1 = 1$, $\lambda_2 = 4$.

Hence from the equations (2), we get the relations $\pm 1 = a_0 + a_1$ and

$\pm 2 = a_0 + 4a_1$

Solving these relations we obtain the values

$a_0 = \pm\frac{2}{3}$, $a_1 = \pm\frac{1}{3}$

$\sqrt{A} = \pm\frac{2}{3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \pm\frac{1}{3}\begin{pmatrix} 1 & 1 \\ 0 & 4 \end{pmatrix}$.

$= \pm\frac{1}{3}\begin{pmatrix} 3 & 1 \\ 0 & 6 \end{pmatrix} = \pm\begin{pmatrix} 1 & 1/3 \\ 0 & 2 \end{pmatrix}$.

5. Conclusions

From the discussions it is seen that every matrix has no square root. For some matrices there exists two square roots **,**some matrices have four square roots, while some matrices may have infinite square roots. The methods employed in this paper to find square roots of matrices are easiest and simple one. This method can be extended to find square roots for 3× 3 matrices also.

7. References

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