Normal-Doubly truncated Normal Stochastic Production Frontier Model

S. Kannaki\textsuperscript{1} & Mary Louis\textsuperscript{2}

\textsuperscript{1} Ph.D Research Scholar, Department of Mathematics, Avinashilingam university, Coimbatore-641108, Tamil Nadu, India
\textsuperscript{2} Associate Professor of Mathematics, Faculty of Engineering, Avinashilingam University, Coimbatore-641108, Tamil Nadu, India

**Abstract:** Stochastic Production Frontier Analysis (SPFA) is a Mathematical modelling used to estimate individual efficiency scores in the field of production. The purpose of Stochastic Production Frontier Analysis (SPFA) is to measure how efficient a producer is with given observations of input and output by using two error terms, $u_i$ and $v_i$. In this paper, the technical efficiency, \( TE_i = \exp(-u_i) = \exp \left( -E \left( \frac{\theta_i}{\epsilon} \right) \right) \) of normal- doubly truncated stochastic production frontier model was derived. The parameters were evaluated using Maximum likelihood Estimates.

1. Introduction

Stochastic production frontier analysis (SPFA) is an exciting method of economic production modeling that is relevant to hospitals, agriculture, stock markets, manufacturing factories, and services etc. The stochastic frontier analysis was first proposed by Aigner et al. (1977) and Meeusen and van de Broeck (1977) and Battese and Corra (1977). The most important purpose of stochastic frontier analysis is to estimate technical efficiencies of firms based on their choices of input combinations, price of inputs and the level of outputs produced. Based on the estimates of technical efficiencies, the rank of firms in terms of efficiency can be obtained.

The main idea of stochastic production frontier analysis is the introduction of composite error term which contain two components. The first error component $u$ represents the unmeasured variables such as weather, walkout, epidemic, and other variables which are undefined in the production function and also $u$ is intended to capture the technical inefficiency. The second error component $v$ is intended to capture the effects of statistical noise, or exogenous shocks beyond the control of the firms which is identically and independently normal distributed with mean 0 and variance $\sigma_v^2$. The technical inefficiency term $u$ follows various continuous distributions. Distributional assumption plays a vital role in measuring the technical efficiency of each producer. Meeusen and van den Broeck (1977) assigned an exponential distribution, Battese and Corra (1977) assigned a half normal distribution, Aigner, Lovell and Schmidt (1977) assigned both exponential and half normal to $u$. Greene (1990) proposed a Gamma distribution and Stevenson (1980) proposed Gamma and truncated normal distributions. In this paper, doubly truncated normal distribution is assigned for $u$.

Koopmans (1951) provided a definition of technical efficiency “Achieving maximum output from a given input vector”. In stochastic production frontier analysis, The technical efficiency of producer is $TE_i = \frac{y_i}{f(x_i, \beta)} \exp \left\{ v_i \right\}$ which defines technical efficiency as the ratio of observed output to the maximum feasible output, conditional on $\exp \left\{ v_i \right\}$. (Kumbhakar and Lovell, 2003).

Generally Technical efficiency, TEi, can be attained by the exponential conditional expectation of u given the composed error term $\epsilon$, which is given by $TE_i = \exp \left( -E \left( \frac{u_i}{\epsilon} \right) \right)$ (Jondrow et al., 1982). For estimating the parameters in SPFA, Maximum likelihood Estimate (MLE) method was used (Aigner and Chu, 1968).

**II. NORMAL-DOUBLY TRUNCATED NORMAL STOCHASTIC PRODUCTION FRONTIER MODEL (NDTNSPFM)**

The Stochastic Production Frontier Model can be expressed as $y = f(x, \beta) e^{v-u}$

For the estimation of Stochastic Production Frontier Model, the following assumptions in the distribution were made:

1. $v \sim iid \ N(0, \sigma_v^2)$
2) \( \mathbf{u} \sim \text{iid} \) Doubly Truncated Normal distribution in the interval \((0, B)\)

3) \( \mathbf{u} \) and \( \mathbf{v} \) are distributed independently of each other and of the repressors.

The probability density function of \( v \) is given by

\[
f(v) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{v^2}{2\sigma^2}}, -\infty < v < \infty \quad \text{(1)}
\]

The probability density function of \( u \geq 0 \) is given by

\[
f(u) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(u-u_-)^2}{2\sigma^2}}, 0 < u < B > 0
\]

Where \( \Phi \left( \frac{B - \mu}{\sigma_u} \right), \Phi \left( \frac{-\mu}{\sigma_u} \right) \) are distribution function of a standard normal variable, \( \phi \left( \frac{u - \mu}{\sigma_u} \right) \) is a probability density function of a standard normal variable, \( 1_{[0, B]}(u) \) is an indicator function and \( B \) is a truncation parameter.

Since \( u \) and \( v \) are independent, the joint density function of \( u \) and \( v \) is

\[
f(u, v) = f(u) f(v)
\]

\[
= \frac{1}{2\pi \sigma_u \sigma_v} \left[ \Phi \left( \frac{B - \mu}{\sigma_u} \right) - \Phi \left( \frac{-\mu}{\sigma_u} \right) \right] \quad \text{(3)}
\]

Making the transformation, \( \varepsilon = v - u \Rightarrow v = u + \varepsilon \). The joint density function of \( u \) and \( \varepsilon \)

\[
f(u, \varepsilon) = \frac{1}{2\pi \sigma_u \sigma_v} \left[ \Phi \left( \frac{B - \mu}{\sigma_u} \right) - \Phi \left( \frac{-\mu}{\sigma_u} \right) \right]
\]

The marginal density function of \( \varepsilon \) is obtained by integrating \( f(u, \varepsilon) \) with respect to \( u \).

\[
f(\varepsilon) = \int_0^B f(u, \varepsilon) \, du
\]

\[
= \int_0^B \frac{1}{2\pi \sigma_u \sigma_v} \left[ \Phi \left( \frac{B - \mu}{\sigma_u} \right) - \Phi \left( \frac{-\mu}{\sigma_u} \right) \right] \, du \quad \text{(5)}
\]
where $\lambda = \frac{\mu}{\sigma_u}$, $\sigma^2 = \sigma_u^2 + \sigma_v^2$.

\[ f(\varepsilon) = \frac{1}{\sqrt{2\pi}} \left[ \phi \left( \frac{B - \mu}{\sigma_u} \right) - \phi \left( \frac{-\mu}{\sigma_u} \right) \right] \]

Let $u = \frac{t^2}{2}$, $du = t \, dt$.

\[ E(\varepsilon) = E(v - u) = E(v) - E(u) = -E(u) \quad \text{(9)} \]

\[ E(\varepsilon) = -\frac{n}{\sigma_u} \int_{-\infty}^{\infty} u \, f(u) \, du \quad \text{(10)} \]

\[ = -\frac{1}{\sigma_u \sqrt{2\pi} \phi \left( \frac{B - \mu}{\sigma_u} \right) - \phi \left( \frac{-\mu}{\sigma_u} \right)} \left[ \frac{\int \left( \varepsilon^2 \right) f(\varepsilon) \, d\varepsilon}{\sigma_u^2} \right] \]

\[ \text{Var}(\varepsilon) = \text{Var}(v - u) = \text{Var}(v) + \text{Var}(u) \quad \text{(15)} \]

\[ \text{Var}(u) = E(u^2) - (E(u))^2 \quad \text{(16)} \]

\[ E(u^2) = \int_{0}^{\infty} u^2 \, f(u) \, du \quad \text{(17)} \]
III. MEASURE OF TECHNICAL EFFICIENCY

OF NORMAL-DOUBLY TRUNCATED NORMAL STOCHASTIC PRODUCTION FRONTIER MODEL (NDTNSPFM)

Technical Efficiency,

\[ TE_i = \exp(-\hat{u}) = \exp\left(-E\left(\frac{u}{\sigma}\right)\right) \]

Consider,

\[ f\left(\frac{u}{\sigma}\right) = \frac{f(u, \varepsilon)}{f(\varepsilon)} \]
IV. Estimation of parameters of NDTNSPFM using Maximum likelihood estimation

The likelihood function of the sample is the product of the density function of the individual observations, which is given as, 

\[ L(\text{sample}) = \prod_{i=1}^{N} f(\epsilon_i) \]  

The Log likelihood function for a sample of N producers is 

\[ \ln L = -\frac{1}{2} \ln 2\pi - \ln \sigma - \frac{1}{2} \left( \frac{\mu + \epsilon_i}{\sigma} \right)^2 \]

\[ = -\ln \left[ \Phi \left( \frac{B - \mu}{\sigma} \right) \right] - \ln \left[ \Phi \left( \frac{\lambda - \mu}{\sigma} \right) \right] + \ln \left[ \Phi \left( \frac{B + \epsilon_i\lambda - \mu}{\sigma} \right) \right] \]

\[ \text{Parameters } B, \sigma^2, \beta, \lambda, \mu \text{ can be estimated using the first order conditions of the maximization of log-likelihood function.} \]
where

\[ \ln L = \frac{1}{2} \ln 2\pi - \ln \sigma - \frac{1}{2} \left( \frac{y_i - x_i \beta}{\sigma} \right)^2 \]

\[ \ln \left( \frac{(\beta - \mu)}{\sigma} \left( x_i^2 + 1 \right)^{\frac{1}{2}} \right) - \ln \left( \frac{(-\mu)}{\sigma} \left( x_i^2 + 1 \right)^{\frac{1}{2}} \right) \]

\[ \ln \left( \left( B + \mu \right) \phi \left( K_i \right) \right) - \ln \left( \left( -\mu \right) \phi \left( K_i \right) \right) \]

Partially differentiating (33) w.r.t. \( \beta \)

\[ \frac{\partial \ln L}{\partial \beta} = \frac{(e_i + \mu) x_i + \lambda x_i \phi \left( K_i \right) - \phi \left( K_i \right)}{\sigma^2 \phi \left( K_i \right) - \Phi \left( K_i \right)} \]

Partially differentiating (32) w.r.t. \( \sigma^2 \)

\[ \frac{\partial \ln L}{\partial \sigma^2} = \frac{\phi \left( K_i \right) - \Phi \left( K_i \right)}{2\sigma^2 \sigma^2} - \frac{1}{2\sigma^2} \frac{1}{2\sigma^2} \phi \left( K_i \right) - \Phi \left( K_i \right) + \frac{1}{2\sigma^2} \phi \left( K_i \right) - \Phi \left( K_i \right) \]

Partially differentiating (32) w.r.t. \( \lambda \)

\[ \frac{\partial \ln L}{\partial \lambda} = \frac{(e_i + \mu)^2}{\sigma^2} \phi \left( K_i \right) - \Phi \left( K_i \right) + \frac{(e_i + \mu)}{\sigma^2} \phi \left( K_i \right) - \Phi \left( K_i \right) \]

Partially differentiating (32) w.r.t. \( \mu \)

\[ \frac{\partial \ln L}{\partial \mu} = \frac{\phi \left( K_i \right) - \Phi \left( K_i \right)}{\sigma^2} + \frac{1}{\lambda \sigma} \phi \left( K_i \right) - \Phi \left( K_i \right) \]

Partially differentiating (32) w.r.t. \( B \)

\[ \frac{\partial \ln L}{\partial B} = \frac{(e_i + \mu)^3}{\sigma^2} \phi \left( K_i \right) - \Phi \left( K_i \right) + \frac{1}{\lambda \sigma} \phi \left( K_i \right) - \Phi \left( K_i \right) \]

Where

\[ K_1 = \left( \frac{B - \mu}{\sigma} \right) \left( \lambda^2 + 1 \right)^\frac{1}{2}, K_2 = \left( \frac{-\mu}{\sigma} \right) \left( \lambda^2 + 1 \right)^\frac{1}{2} \]

\[ K_3 = \left( B + e_i \right) \lambda + \left( B - \mu \right) \lambda^{-1} \frac{1}{\sigma}, K_4 = \frac{e_i \lambda - \mu \lambda^{-1}}{\sigma} \]

Equating the equations (34), (35), (36), (37), (38) to zero and then solving, the optimum point can be obtained.

**V. Conclusion**

In the present study to measure technical efficiency using normal-doubly truncated normal stochastic production frontier model,

\[ E \left( \mu / \epsilon \right) = \mu_0 + \sigma_0 \left( \frac{-\mu}{\sigma} \right) \left( \frac{B - \mu}{\sigma} \right) \]

is derived. The model derived can be used to measure technical efficiency scores of individual firms and also can be applied to identify factors which affect the production. The best firm can be identified and the average production can be increased by adopting the best practices followed by the model firm.

---

**References**


