Abstract: In this paper, we present an algorithm to find a minimum spanning tree of fuzzy graph \( G \), and also for calculating the value of minimum spanning Steiner fuzzy tree of \( G \), when costs \( c_{ij} \) are associated with its edges. The algorithm uses random sampling in combination with Steiner points for verifying a minimum spanning Steiner fuzzy tree problem.

Keywords: Shortest Spanning Steiner fuzzy tree (SSSFT), Steiner points, rectilinear problem.

1. INTRODUCTION
The Shortest Spanning Steiner fuzzy tree (SSSFT) of a fuzzy graph has various applications in cases where roads (gas pipelines, electric power lines, etc) are to be used to connect “n” points together in such a way as to minimize the total length of road that has to be constructed. If the “n” points to be connected are on a Euclidean plane, they can be represented as vertices of a complete fuzzy graph \( G \) with arc costs being the straight line distances between the corresponding end points. The Shortest Spanning Steiner fuzzy tree (SSSFT) of \( G \) is then (provided no road junction outside the given “n” points are allowed) the required minimum –cost road network. If junctions outside the given “n” points are allowed, then an even shorter road network may be possible and finding it is a problem known as Fuzzy Steiner’s Problem.

While a spanning fuzzy tree spans all vertices of a given fuzzy graph, a Steiner fuzzy tree spans a given subset of vertices. In the Steiner minimal fuzzy tree problem, the vertices are divided into two parts: terminals and non-terminal vertices. The terminals are the given vertices which must be included in the solution. The cost of a Steiner fuzzy tree is defined as the total edge weight. A Steiner fuzzy tree may contain some non-terminal vertices to reduce the cost. Let \( V \) be a set of vertices. In general, we are given a set \( L \subset V \) of terminals and a metric defining the distance between any two vertices in \( V \). The objective is to find a connected fuzzy sub-graph spanning all the terminals of minimal total cost. Since the distances are all nonnegative in a metric, the solution is a fuzzy tree structure.

2. The Steiner Problem
The Steiner fuzzy network problem is the optimization problem of finding a fuzzy tree \( F_T = (X', E') \) in \( G \) such that \( X \subset X' \) and the weights \( W(e) \) is as small as possible, given a set \( X \) of vertices in an undirected graph \( G = (X, E) \) with a non-negative weight function \( W \) defined on \( E \). The fuzzy tree \( F_T \) is a Steiner fuzzy tree of the set \( X \). The vertices in \( X' - X \) are the Steiner points of \( X \) with respect to \( T \).

Definition: 1
A Steiner fuzzy tree with respect to a pair of vertices in an undirected fuzzy network is any shortest fuzzy path \( P \) between them, and the Steiner points of this pair with respect to \( P \) are the intermediate vertices of this fuzzy path.

Example: 1
Considering the 4 points shown in Fig. 1(a), the shortest spanning fuzzy tree is as shown, whereas the introduction of two new points \( S_1 \) and \( S_2 \) in the middle, produces a shortest spanning fuzzy tree spanning all 6 points, whose total length is less than that of the previous shortest spanning fuzzy tree shown in Fig.1(b).

Thus, in the original Steiner problem, as many Steiner points as necessary could be added anywhere in the fuzzy graph, in order to produce the shortest fuzzy tree spanning the specified set of \( n \) points. This resulting spanning fuzzy tree is then called a shortest Steiner fuzzy tree.
2.1 PROPERTIES OF THE SHORTEST STEINER FUZZY TREE

- For a Steiner point $S_i$, the degree $d(S_i) = 3$. It can be easily shown by geometrical considerations that the angle between links incident at any Steiner point must be $120^\circ$, and the angle between the two links incident at any Steiner point $S_i$. This point is, therefore, the "centre" (Steiner centre) of an imaginary triangle whose vertices are the other 3 points, to which $S_i$ is linked in the shortest Steiner fuzzy tree. Some of the points forming the vertices of this triangle may themselves be other Steiner points. In Fig 1(b), Steiner points $S_2$ is the Steiner centre of the imaginary triangle with vertices $x_3$, $x_4$ and $S_i$.

For a vertex $x_i \in X$, $d(x_i) \leq 3$. If $d(x_i) = 3$, then the angle between any two of the three links incident at $x_i$ must be $120^\circ$, and if $d(x_i) = 2$, the angle between the two links must be greater than or equal to $120^\circ$.

- The number $K$ of Steiner points in a shortest Steiner fuzzy tree is $0 \leq K \leq n-2$, where $n = |X|$.

2. Algorithm for minimum spanning fuzzy tree enumeration

Step 1: The input is a set $Q$ of $m$ vertices in a connected fuzzy network $G = (X,E)$ of order $n$ in which a nonnegative weight is defined on each edge.

Step 2: Construct the complete fuzzy network $G' = (X,E')$ in which the weight of an edge joining two vertices is the shortest distance between them.

Step 3: Let $f = |S \subseteq (X-Q) : |S| \leq (m-2)|$. For each subset $S$ in $f$, find an minimum spanning fuzzy tree of the fuzzy sub graph of $G'$ induced by $Q \cup S$ on $G'$. Among the fuzzy trees thus obtained, select a fuzzy tree $F_t$ of minimum weight.

Step 4: Construct a fuzzy tree $F_T$ from $F_t$, by replacing each edge joining two vertices by the set of edges in a shortest fuzzy between them. Those vertices of $F_T$ that are not in $Q$ will form a set of Steiner points for $Q$.

Example: 2.1

Find the Steiner trees with respect to $V_1 = \{x_3, x_6, x_7\}$ in the fuzzy network.
The shortest distance matrix for this fuzzy network $[D(F_N)]$ is Table-1

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Since $m = 3$, $S$ is any subset of $\{x_1, x_2, x_4, x_5\}$ with at most one element. There are five choices for $(Q \cup S)$:

- $Q_1 = \{x_3, x_6, x_7\}$
- $Q_2 = \{x_3, x_6, x_7, x_1\}$
- $Q_3 = \{x_3, x_6, x_7, x_2\}$
- $Q_4 = \{x_3, x_6, x_7, x_4\}$
- $Q_5 = \{x_3, x_6, x_7, x_5\}$

The minimum weight spanning fuzzy trees of the fuzzy sub graphs of $G'$ induced by these sets have weights 0.09, 0.09, 0.09, 0.12 and 0.08 respectively. So we take the fuzzy sub graph induced by $V_5$ on the complete fuzzy network $G'$. In this fuzzy sub-graph, a minimum spanning fuzzy tree $F_t$ as shown in Fig.4(a).

The weight of the edge in $F_t$ between $x_3$ and $x_5$ is 0.04. This shortest fuzzy path is $x_3 \rightarrow x_2 \rightarrow x_1 \rightarrow x_6$. Thus the minimum Steiner fuzzy tree with respect to $\{x_3, x_6, x_7\}$ is as shown in Fig.4(b). The Steiner points are $x_1, x_2, x_4$. 

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**Fig.3 Fuzzy Network**

**Table 1**

![Table 1](image1)

**Fig. 4(a) Minimum spanning fuzzy tree $F_t$**

![Fig. 4(a) Minimum spanning fuzzy tree $F_t$](image2)
and $x_5$.

**Example 2.2** (6-point rectilinear problem)

Let a fuzzy graph $G$ be found so that the set $X$ of its vertices is the set of distinct grid line intersections and the links of $G$ correspond to grid lines joining two intersection points. Fig. 5(b) shows an example of a shortest Steiner fuzzy tree of a 6-point rectilinear problem and, for comparison, fig. 5(a) shows the shortest spanning fuzzy tree of this problem.
Conclusion

The minimum weight spanning fuzzy trees of the fuzzy sub graphs of G' induced by the set \{x_3, x_6, x_7, x_5\} having weight 0.08. So we take the fuzzy sub graph induced by V_3 on the complete fuzzy network. Thus the minimum Steiner fuzzy tree with respect to \{x_3, x_6, x_7\} is as shown in Fig.4(b). The Steiner points are x_1, x_2 and x_5. Also, the value of the shortest spanning fuzzy tree of 6-point rectilinear problem is 0.18 and the value of the shortest spanning Steiner fuzzy tree of 6-point rectilinear problem is 0.15. Hence we conclude that the distance or cost of transportation problem or optimization problem are minimized by Steiner points, and are useful to reduce the time or money. However, these algorithms are computationally inefficient procedures (for more than 10 vertices) even though they are much better than the complete enumeration of shortest spanning fuzzy tree's of all fuzzy sub-graphs G' of G.

References

[1] Hakimi,S.L.(1971). Steiner’s problem in graphs and its implications, Networks,1,p.113