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Function Spaces for Axisymmetric Solenoidal Vector Fields

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***Abstract**: In this paper, we study the three-dimensional axisymmetric Navier-Stokes system with nonzero swirl. By establishing a new key inequality for the pair (ω^r/r,ω^θ/r), we get several Prodi-Serrin type regularity criteria based on the angular velocity, u^(θ). Furthermore, we also get several Prodi-serrin type regularity criteria based on one component of the solutions, say ω^3 or u^3. We consider the vorticity-stream formulation of axisymmetric incompressible flows and its equivalence with the primitive formulation and some of its properties were studied.*

***Keywords**: Navier-Stokes equations, Regularity Criteria, Global Well-posedness, Axisymmetric flow, Euler equation, Pole condition, Leray Solution.*

1. INTRODUCTION

Consider the initial value problem of 3D Navier-Stokes equations

$$\begin{cases} \partial_t u + (u.\nabla)u - \Delta u + \nabla p = 0, (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \nabla . u, \\ u|_{t=0}, \end{cases}$$

where$u(t,x) = (u^1, u^2, u^3), p(t,x)$ and u_0 denote respectively, te fluid velocity field, the pressure, and the given initial velocity field.

Axisymmetric flow is an important subject in fluid dynamics and has become standard textbook material as a starting point of theoretical study for complicated flow patterns.

2.PRELIMINARIES

2.1 Definition

A fluid is defined as either a gas or a liquid. **Fluid mechanics** is the study behavior of liquids and gasses. More properly defined fluid mechanics is the study of fluids and forces on them.

$$\nabla . v = 0$$

2.2 Definition

Navier-stokes equation

$$\rho\left(\frac{\partial u}{\partial t} + u \cdot \nabla u\right) = -\nabla p + \nabla \cdot T_D + f$$

$T_D =$ Deviatoric stress tensor.

$f =$ volume density of the body forces acting on the fluid.

∇here is the del operator.

2.3 Definition

Euler equation

$$\frac{\partial p}{\partial t} + \nabla \cdot (\rho u) = 0$$

$\rho =$fluid mass density

$u =$The flow velocity vector.

2.4 Definition

Axis of rotation is a line about which a three-dimensional object is rotated in space.

Axis of rotation of an object can be within the body or outside the body.

2.5 Theorem

Let $u \in C([0,T): H^2(\mathbb{R}^3)) \cap L^2_{loc}([0,T): H^3(\mathbb{R}^3))$ be the unique axisymmeric solution of the Navier - stokes equations with the axisymmetric initial data $u_0 \in H^2(\mathbb{R}^3)$ and $div\, u_0 = 0$. Assume

$$\left\| r^d u^\theta \right\|_{L^\infty([0,t); L^{\frac{3}{1-d}}(\mathbb{R}^3))} \le \frac{1}{C_0},$$

whereC_0 is a constant and $t < T^*$. Then, we have

$$\|\Phi(t)\|_2^2 + \|\Gamma(t)\|_2^2 \le \|\Phi_0\|_2^2 + \|\Gamma_0\|_2^2 \quad (1)$$

Proof:

From the Sobolev - Hardy inequality

$$\frac{1}{2}\frac{d}{dt}(\|\Phi\|_2^2 + \|\Gamma\|_2^2) + \left(\left\|\widetilde{\nabla}\Phi\right\|_2^2 + \left\|\widetilde{\nabla}\Gamma\right\|_2^2\right) \le I_1 + I_2 + 2I_3 \quad (2)$$

(1) and (2), we have a priori estimate on $(0,t)$, that

$$\frac{1}{2}\frac{d}{dt}(\|\Phi\|_2^2+\|\Gamma\|_2^2)+\|\tilde{\nabla}\Phi\|_2^2+\|\tilde{\nabla}\Gamma\|_2^2$$
$$\leq I_1+I_2+2I_3$$
$$\leq C_0\|r^d u^\theta\|_{\frac{3}{1-d}}\|\tilde{\nabla}\Gamma\|_2\|\tilde{\nabla}\Phi\|_2 \quad (3)$$
$$\leq \frac{1}{2}\|\tilde{\nabla}\Gamma\|_2^2+\frac{1}{2}\|\tilde{\nabla}\Phi\|_2^2$$

Which implies (1).

Hence the proof

2.6 Theorem

Under the condition in theorem (2.5), then there exists a positive constant C_1, such that,

$$\|\omega^\theta\|_{L^\infty((0,t);L^2(\mathbb{R}^3))}^2$$
$$\leq \|\omega^\theta\|_2^2+C_1(\|\Gamma_0\|_2+\|\Phi_0\|_2)^{\frac{4}{3}}\|u_0\|_2^2 \quad (4)$$

Proof:

Multiplying the equation (1) by ω^θ, and integrating the resulting equations over $\mathbb{R}^3$ respectively, using the integration by parts rule, Holder's inequality and the sobolev embedding Theorem, we have

$$\frac{1}{2}\frac{d}{dt}\|\omega^\theta\|_2^2+\left(\|\tilde{\nabla}\omega^\theta\|_2^2+\|\Gamma\|_2^2\right)$$
$$=\int_{\mathbb{R}^3}(\Gamma u^r-2\Phi u^\theta)\omega^\theta\,dx$$
$$\leq 2(\|\Gamma\|_2\|u^r\|_6+\|\Phi\|_2\|u^\theta\|_6)\|\omega^\theta\|_3$$
$$\leq C(\|\Gamma\|_2+\|\Phi\|_2)\|\nabla u\|_2\|\omega^\theta\|_2^{\frac{1}{2}}\|\tilde{\nabla}\omega^\theta\|_2^{\frac{1}{2}}$$
$$\leq C(\|\Gamma\|_2+\|\Phi\|_2)^{\frac{4}{3}}\|\nabla u\|_2^{\frac{4}{3}}\|\omega^\theta\|_2^{\frac{2}{3}}+\frac{1}{2}\|\tilde{\nabla}\omega^\theta\|_2^2$$
$$\leq C(\|\Gamma\|_2+\|\Phi\|_2)^{\frac{4}{3}}\|\nabla u\|_2^2+\frac{1}{2}\|\tilde{\nabla}\omega^\theta\|_2^2. \quad (5)$$

From equation $\frac{1}{2}\|u\|_2^2+\int_0^t\|\nabla u\|_2^2 d\leq\frac{1}{2}\|u_0\|_2^2$, for all $t\geq 0$ and (4), we get (4).

Hence the proof

3. REGULARITY OF 3D AXISYMMETRIC NAVIER-STOKES EQUATION

3.1 Theorem

Let$u=u_x e_x+u_r e_r+u_\theta e_\theta\in C^k(R^3,R^3), k\geq 0$, then for any fixed $\theta\in[0,\pi), u_x(.,.,\theta), u_r(.,.,\theta), u_\theta(.,.,\theta)\in C^k(R\times\overline{R^+})$. Moreover,

$$\partial_r^j u_x(x,0^+,\theta)=(-1)^j\partial_r^j u_x(x,0^+,\theta+\pi),\quad 0\leq j\leq k, \quad (1)$$
$$\partial_r^j u_r(x,0^+,\theta)=(-1)^{j+1}\partial_r^j u_r(x,0^+,\theta+\pi),\quad 0\leq j\leq k, \quad (2)$$
$$\partial_r^j u_\theta(x,0^+,\theta)=(-1)^{j+1}\partial_r^j u_\theta(x,0^+,\theta+\pi),\quad 0\leq j\leq k.(3)$$

Proof

Let $u=u_x(x,r,\theta)e_x+u_r(x,r,\theta)e_r+u_\theta(x,r,\theta)e_\theta$. Note that e_x is smooth vector field while e_r and e_θ are discontinuous at the axis of rotation. More specifically, on the cross section

$$z=0, y>0,$$

We have

$e_x(x,y,z=0)=e_x(x,r=|y|,\theta=0), e_x(x,-y,z=0)=e_x(x,r=|y|,\theta=\pi),$ (4)$e_y(x,y,z=0)=e_r(x,r=|y|,\theta=0), e_y(x,-y,z=0)=-e_x(x,r=|y|,\theta=\pi),$ (5)$e_z(x,y,z=0)=e_\theta(x,r=|y|,\theta=0),$ $e_z(x,-y,z=0)=-e_\theta(x,r=|y|,\theta=\pi)$ (6)

Consequently

$u_x(x,y,z=0)=u_x(x,r=|y|,\theta=$ *0, uxx,−y,z=0=uxx,r=y,θ=π,* (7)

$u_y(x,y,z=0)=u_r(x,r=|y|,\theta=$ *0, uyx,−y,z=0=−uxx,r=y,θ=π* (8)

$u_z(x,y,z=0)=u_\theta(x,r=|y|,\theta=$ *0, uzx,−y,z=0=−uθx,r=y,θ=π*(9)

Taking the limit $y\to 0^+$, it follows that (1-3) holds with $\theta=0$.

Hence the proof

3.2 Theorem

Let $u\in C_s^k(R^3,R^3), k\geq 0$, be represented by $u=ue_\theta+\nabla\times(\psi e_\theta)$ with $u\in C_s^k(R\times\overline{R^+})$and $\psi(x,r)\in C^{k+1}(R\times\overline{R^+})$.

Then $(\nabla\times)^\ell \mathrm{u}\in C_s^{k-\ell}(R^3,R^3)$ and

$(\nabla\times)^{2\mathrm{m}}\mathrm{u}=(-1)^{\mathrm{m}}\left((\mathcal{L}^m u)e_\theta+\nabla\times\right.$ *ℒmueθ,*if *2m≤k,*

$$(\nabla \times)^{2m+1} u = (-1)^{m+1}\left((\mathcal{L}^{m+1}\psi)e_\theta + (-1)^m \nabla \times \left((\mathcal{L}^m u)e_\theta\right)\right),$$

if$2m+1 \le k$,

Where

$$\mathcal{L} = \nabla^2 - \frac{1}{r^2} = \left(\partial_r^2 + \frac{1}{r}\partial_r + \partial_x^2\right) - \frac{1}{r^2}$$

Moreover

$$\mathcal{L}^m u \in C_s^{k-2m}\left(R \times \overline{R^+}\right), \quad if \quad 2m \le k,$$

$$\mathcal{L}^{m+1}\psi \in C_s^{k-1-2m}\left(R \times \overline{R^+}\right), \quad if \quad 2m+1 \le k.$$

Proof

For any $\phi \in C_s^i\left(R \times \overline{R^+}\right)$, we have $\phi e_\theta \in C_s^i$ from

Theorem 3.2.

$$\nabla \times u = \frac{1}{r}\begin{vmatrix} e_x & e_r & re_\theta \\ \partial_x & \partial_r & \partial_\theta \\ u_x & u_r & ru_\theta \end{vmatrix} \qquad (10)$$

With a straight forward calculation using (32), it is easy to verify that for $i \ge 2$,

$$\nabla \times \nabla \times (\phi e_\theta) = -(\mathcal{L}\phi)e_\theta. \qquad (11)$$

On the other hand, it is clear that

$$\nabla \times \nabla \times (\phi e_\theta) \in C_s^{i-2},$$

and therefore from theorem 3.2 (a),

$$\mathcal{L}\phi \in C_s^{i-2}\left(R \times \overline{R^+}\right). \qquad (12)$$

The theorem follows from (33) and (34).

Hence the proof

3.3 Theorem

If $v \in C^k\left(R \times \overline{R^+}\right)$ and $v(x, 0^+) = 0$,

Then

$$\lim_{r\to 0^+} j\partial_r^{j-1}\left(\frac{v(x,r)}{r}\right) = \partial_r^j v(x,0^+), \quad 1 \le j \le k. \qquad (13)$$

Proof:

Since $v \in C^k\left(R \times \overline{R^+}\right)$, we have

$$\psi(x,r) = a_1(x)r + a_2(x)r^2 + \cdots + a_{k-1}(x)r^{k-1} + R_k(v) \qquad (14)$$

from Taylor's Theorem. Here

$$a_\ell(x) = \frac{1}{\ell!}\partial_r^\ell \psi(x,0^+),$$

$$R_k(v) = \int_0^r \partial_r^k v(x,s)\frac{(r-s)^{k-1}}{(k-1)!}ds$$

and

$$\partial_r^\ell R_k(v)(x,0^+) = 0, \quad 0 \le \ell \le k-1,$$

$$\partial_r^k R_k(v)(x,0^+) = \partial_r^k v(x,0^+). \qquad (15)$$

From (14), it follows that

$$\partial_r^{k-1}\left(\frac{v(x,r)}{r}\right) = \partial_r^{k-1}\left(\frac{R_k(v)}{r}\right)$$

$$= \sum_{\ell=0}^{k-1} C_{k-1}^\ell (-1)^\ell \ell! \frac{\partial_r^{k-\ell-1}R_k(v)}{r^{\ell+1}} \qquad (16)$$

The assertion (15) is obvious for $j < k$.For $j = k$, from (15), (16) and l'Hospital's rule, we can easily derive

$$\lim_{r\to 0^+}\partial_r^{k-1}\left(\frac{v(x,r)}{r}\right) = \left(\sum_{\ell=0}^{k-1} C_{k-1}^\ell (-1)^\ell \frac{1}{\ell+1}\right)\partial_r^k v(x,0^+) = \frac{1}{k}\partial_r^k v(x,0^+).$$

Hence the proof

3.4 Theorem

If $v \in C^{2m}(R \times \overline{R^+}) \cap C^{2m-2}(R \times \overline{R^+})$

Then $\partial_r^{2m-1}\mathcal{L}v(.,0^+) = 0$ if and only if $\partial_r^{2m}v(.,0^+) = 0$ (17)

Proof:

Since

$$\left(\nabla^2 - \frac{1}{r^2}\right)v = \left(\partial_x^2 v + \partial_r^2 v + \partial_r\left(\frac{v}{r}\right)\right),$$

one has

$$\partial_r^{2m-2}\mathcal{L}v = \left(\partial_x^2\partial_r^{2m-2}v + \partial_r^{2m}v + \partial_r^{2m-1}\left(\frac{v}{r}\right)\right),$$

It follows from Theorem 3.3 that

$$\partial_r^{2m-2}\mathcal{L}v(x,0^+) = \frac{2m+1}{2m}\partial_r^{2m}v(x,0^+)$$

Hence the proof

3. AXISYMMETRIC NAVIER-STOKES EQUATIONS AND EQUIVALENCE RESULTS

4.1 Theorem

If $2m \leq k-2$ and

$$\psi \in C^{k+1}\left(R \times \overline{R^{+}}\right) \cap C_s^{2m}\left(R \times \overline{R^{+}}\right),$$
$$u \in C^{k}\left(R \times \overline{R^{+}}\right) \cap C_s^{2m}\left(R \times \overline{R^{+}}\right),$$
$$\omega \in C^{k-1}\left(R \times \overline{R^{+}}\right) \cap C_s^{2m}\left(R \times \overline{R^{+}}\right). \qquad (1)$$

then all the nonlinear terms

$$\partial_t u + u_x \partial_x u + u_r \partial_r u + \frac{u_r}{r} u = \nu \mathcal{L} u,$$
$$\partial_t \omega + u_x \partial_x \omega + u_r \partial_r \omega - \frac{u_r}{r}\omega = \frac{1}{r}\partial_x(u^2) + \nu \mathcal{L}\omega,$$
$$\omega = -\mathcal{L}\psi. \qquad (2)$$

in (1).

$$u_x \partial_x \omega, \quad u_r \partial_r \omega, \quad \frac{u_r}{r}\omega, \quad \frac{1}{r}\partial_x(u^2) \qquad (3)$$

and

$$u_x \partial_x u, \quad u_r \partial_r u, \quad \frac{u_r}{r} u, \qquad (4)$$

are in$C_s^{2m}\left(R \times \overline{R^{+}}\right)$.

Proof

"Let (ψ, u, ω) be a solution to the axisymmetric Euler equation (1) in the class

$$\begin{pmatrix} \psi(t,x,r) \\ u(t,x,r) \\ \omega(t,x,r) \end{pmatrix} \in C^0\left([0,T), \begin{pmatrix} C^{k+1}(R \times \overline{R^{+}}) \\ C^{k}(R \times \overline{R^{+}}) \\ C^{k-1}(R \times \overline{R^{+}}) \end{pmatrix}\right)$$

with$k \geq 2$ and

$$u = \nabla \times (\psi e_0) + u e_0$$

Then for $0 < t < T,\ 0 \leq j \leq k,$

$u(t,\cdot) \in C_s^{j}(R^3, R^3)$if and only if$u(0,\cdot) \in C_s^{j}(R^3, R^3)$"

By the above statement and (2) holds true under the first two condition in (21). hence, ∂_r^{2m} of four terms in (3) at $r = 0^{+}$ is given by (13-16) with $v(t,x) = \partial_r^{2m}\omega(t,x,0^{+})$.

The third condition in (2) implies that $v = 0$.

Hence, one has

$$\partial_r^{2m}(u_x \partial_x \omega)|_{r=0^{+}} = \partial_r^{2m}(u_r \partial_r \omega)|_{r=0^{+}}$$
$$= \partial_r^{2m}((\frac{u_r}{r})\omega)\Big|_{r=0^{+}}$$
$$= \partial_r^{2m}(\frac{1}{r}\partial_x(u^2))|_{r=0^{+}}$$
$$= 0$$

Obviously, for $j < m$, one also has

$$\partial_r^{2j}(u_x \partial_x \omega)|_{r=0^{+}} = \partial_r^{2j}(u_r \partial_r \omega)|_{r=0^{+}}$$
$$= \partial_r^{2j}((\frac{u_r}{r})\omega)\Big|_{r=0^{+}}$$
$$= \partial_r^{2j}(\frac{1}{r}\partial_x(u^2))|_{r=0^{+}}$$
$$= 0$$

Using exactly same argument. This implies that

$$u_x \partial_x \omega, \quad u_r \partial_r \omega, \quad \frac{u_r}{r}\omega, \quad \frac{1}{r}\partial_x(u^2) \in C_s^{2m}\left(R \times \overline{R^{+}}\right). \qquad (5)$$

Next, (4) holds true under the three conditions in (2).

Hence, ∂_r^{2m} of three terms in (5) at $r = 0^{+}$ is given by (18-20) with$v(t,x) = \partial_r^{2m} u(t,x,0^{+})$.

The second condition in (2) implies that $v = 0$. Hence, we used ∂_r^{2m} instead of ∂_r^{2m+2} in (18-20). Hence, one has

$$\partial_r^{2m}(u_x \partial_x u)|_{r=0^{+}} = \partial_r^{2m}(u_r \partial_r u)|_{r=0^{+}}$$
$$= \partial_r^{2m}((\frac{u_r}{r})u)\Big|_{r=0^{+}}$$
$$= 0$$

Again, for $j < m$, one also has

$$\partial_r^{2j}(u_x \partial_x u)|_{r=0^{+}} = \partial_r^{2j}(u_r \partial_r u)|_{r=0^{+}}$$
$$= \partial_r^{2j}((\frac{u_r}{r})u)\Big|_{r=0^{+}}$$
$$= 0$$

This implies that

$$u_x \partial_x u, \quad u_r \partial_r u, \quad \frac{u_r}{r}u, \in C_s^{2m}\left(R \times \overline{R^{+}}\right). \qquad (6)$$

Hence the proof

4.2 Theorem

Let $v \in C^1(R^3, R^3)$, $\nabla \cdot v = 0$, then there exists a $v^{sym} \in C_s^1(R^3, R^3)$ and

$$\overline{v_x}(x,r,\theta) = v_x^{sym}(x,r),$$

$$\overline{v_r}(x,r,\theta) = v_r^{sym}(x,r),$$

$$\overline{v_\theta}(x,r,\theta) = v_\theta^{sym}(x,r), \qquad (7)$$

where

$$\bar{f}(x,r) = \frac{1}{2\pi}\int_0^{2\pi} f(x,r,\theta)d\theta$$

Proof

Since $v \in C^1(R^3,R^3)$, $\nabla \cdot v = 0$, there exists $\phi = \phi_x e_x + \phi_r e_r + \phi_\theta e_\theta \in C^2(R^3,R^3)$, such that $\nabla \times \phi = v$.

we then define

$$v^{sym} = \nabla \times (\overline{\phi_\theta} e_\theta) + v_\theta^{sym} e_\theta, \quad v_\theta^{sym} = \partial_x \overline{\phi_r} - \partial_r \overline{\phi_x}.$$

It follows that v^{sym} is divergence free and satisfies (5). In addition,

$\phi_x(\cdot,\ \cdot,\theta), \phi_r(\cdot,\ \cdot,\theta), \phi_\theta(\cdot,\ \cdot,\theta) \in C^k\left(R \times \overline{R^+}\right)$for any fixed θ in the view of the below statement

"Let $u \in C^k(R^3,R^3)$, be an axisymmetric vector field,

$$u = u_x(x,r)e_x + u_r(x,r)e_r + u_\theta(x,r)e_\theta.$$

Then $u_x, u_r, u_\theta \in C^k\left(R \times \overline{R^+}\right)$ and

$$\partial_r^{2\ell+1} u_x(x,0^+) = 0, \ \ 1 \le 2\ell + 1 \le k, \qquad (8)$$

$$\partial_r^{2m} u_r(x,0^+) = \partial_r^{2m} u_\theta(x,0^+) = 0, \ \ 0 \le 2m \le k, \qquad (9)$$

We therefore conclude from Bounded convergence Theorem that

$$\lim_{r\to 0^+} \frac{1}{2\pi}\int_0^{2\pi} \partial_x^i \partial_r^j \begin{pmatrix} \phi_x(x,r,\theta) \\ \phi_r(x,r,\theta) \\ \phi_\theta(x,r,\theta) \end{pmatrix} = \frac{1}{2\pi}\int_0^{2\pi} \lim_{r\to 0^+} \partial_x^i \partial_r^j \begin{pmatrix} \phi_x(x,r,\theta) \\ \phi_r(x,r,\theta) \\ \phi_\theta(x,r,\theta) \end{pmatrix},$$

$$0 \le i + j \le 2 \qquad (10)$$

In other words, $\overline{\phi_x},\ \overline{\phi_r},\ \overline{\phi_\theta} \in C^2(R \times \overline{R^+})$. Moreover, (9,10) imply that $\overline{\phi_\theta} \in C_s^2\left(R \times \overline{R^+}\right), v^{sym} \in C_s^1(R^3,R^3)$ therefore $v^{sym} \in C_s^1$.

Hence the proof

5.CONCLUSION

In this paper, the attempt has been made to study about the fulid mechanics. We discussed some theorems onfunction spaces for axisymmetric solenoidal vector fields and we have some ideas about this topic.

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