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An H-ψ Formulation for the Three Dimensional Eddy Current Problem

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***Abstract**: It is a very challenging problem for the direct simulation of the three dimensional eddy currents in grain-oriented (GO) silicon steel laminations since the coating film is only several microns thick over each lamination and the magnetic permeability is nonlinear and anisotropic. In this paper, we studyan H-ψ formulation for the nonlinear eddy current problem.The existence and uniqueness are established for the approximate solution upon introducing proper gauge conditions.*

***Keywords**: Eddy current problem, Nonlinear Maxwell's equations, finite element method, nondestructive evaluation.*

1. INTRODUCTION

Consider the following eddy current problem for magnetic and anisotropic materials in terms of Farady's law and Ampere's law

$$\frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{curl}\, E = 0 \text{in } \mathbb{R}^3,$$

$$curl\, H = J \text{in } \mathbb{R}^3,$$

Where $\boldsymbol{E}$is the electric field, $\boldsymbol{B}$is the magnetic flux, $\boldsymbol{H}$is the magnetic field, and $\boldsymbol{J}$is the current density defined by

$$J = \begin{cases} \sigma E in \Omega_c \\ J_s \ \ in\ \mathbb{R}^3 \setminus \bar{\Omega}_c \end{cases}$$

Definition

i) $\nabla . D = \rho v$
ii) $\nabla . B = 0$
iii) $\nabla \times E = -\frac{\partial B}{\partial t}$
iv) $\nabla \times E = \frac{\partial D}{\partial t} + J$

These are all the **maxwell's equations**.

Theorem

Assume that each connected component of Ω_c is a convex polyhedron and

$$\lim_{\tau \to 0} \left\| H_{s,\tau} - H_s \right\|_{L^2(0,T;L^2(\Omega))} = 0 \quad (1)$$

Where $H_{s,\tau}$ is the piecewise constant interpolation of H_s in time

$$H_{s,\tau}(\cdot\ \ ,t) = H_s(\cdot, t_n) \text{for all } t \in (t_{n-1}, t_n],\ 1 \le n \le N.$$

Then there exists an $H \in L^2(0,T;X)$ such that

$$\lim_{\tau \to 0} H_\tau = \lim_{\tau \to 0} \bar{H}_\tau = H \begin{matrix} strongly\ in\ L^2(0,T;L^2(\Omega\ \ )) \\ weakly\ in\ L^2(0,T;X). \end{matrix}$$

Proof:

To Prove

$$\lim_{\tau \to 0} H_\tau = \lim_{\tau \to 0} \bar{H}_\tau = H \begin{matrix} strongly\ in\ L^2(0,T;L^2(\Omega\ \ )) \\ weakly\ in\ L^2(0,T;X). \end{matrix}$$

Since $L^2(0,T;X)$ is self-reflective. There exist a subsequence of $\{H_\tau\}_{\tau \ge 0}$ and a subsequence of $\{\bar{H}_\tau\}_{\tau \ge 0}$ such that

$$\lim_{\tau \to 0} H_\tau = \lim_{\tau \to 0} \bar{H}_\tau = H \quad weakly\ in\ L^2(0,T;X).$$

Since H can be decomposed into $H = u + \nabla\psi$ where$\psi \in H^1(\Omega)/\mathbb{R}$ and u is the limit of $\bar{u}_\tau$.

Next we prove the strong convergence of $\bar{H}_\tau$.The strong convergence of H_τ comes directly from that of $\bar{H}_\tau$.For convenience we denote the discrete and continuous total magnetic fields by $\hat{H}_\tau = \bar{H}_\tau + H_{s,\tau}$ and $\hat{H} = H + H_s$ respectively.

It follows from (1) and the weak convergence of $\bar{H}_\tau$ that

$\lim_{\tau\to 0}\hat{H}_\tau = \hat{H} \quad weakly\ in\ L^2(0,T;L^2(\Omega)).$ (2)

We deduce that

$(\bar{B}_\tau, \nabla\varphi) = 0$for all $\varphi \in H^1(\Omega)$

We obtain

$$\lim_{\tau\to 0}\int_0^T (\bar{B}_\tau, \hat{H}_\tau - \hat{H}) = \lim_{\tau\to 0}\int_0^T (\bar{B}_\tau, \bar{u}_\tau - u + H_{s,\tau} - H_s) = 0 \quad (3)$$

Noting the monotonicity of $B(\cdot)$ and using (2)-(3), we have

$$\mu_{min} \lim_{\tau\to 0}\|\hat{H}_\tau - \hat{H}\|^2_{L^2(0,T;L^2(\Omega))} \le \lim_{\tau\to 0}\int_0^T (B(\hat{H}_\tau) - B(\hat{H}), \hat{H}_\tau - \hat{H})$$

$$= \lim_{\tau\to 0}\int_0^T (\bar{B}_\tau, \hat{H}_\tau - \hat{H}) - \lim_{\tau\to 0}\int_0^T (B(\hat{H}), \hat{H}_\tau - \hat{H}) = 0.$$

Which shows together with (1) that

$\lim_{\tau\to 0}\|\bar{H}_\tau - H\|^2_{L^2(0,T;L^2(\Omega))} = 0.$

Theorem

Define $\tilde{X}_i = \{v \in H_0(curl, \tilde{\Omega}_i) : div\ v = 0\ in\ \tilde{\Omega}_i\}$ for any $1 \le i \le I$.

Then any function $v \in \tilde{X}$ admits a unique decomposition

$$v = \nabla\psi + \sum_{i=1}^{I} v_i, \quad \psi \in H^1(\Omega)/\mathbb{R}, v_i \in \tilde{X}_i$$

$$\|\nabla\psi\|_{L^2(\Omega)} + \sum_{i=1}^{I}\|v_i\|_{H(curl,\tilde{\Omega}_i)} \le C\|v\|_{H(curl,\Omega)},$$

Where the constant $C > 0$ only depends on $\tilde{\Omega}_1, \dots, \tilde{\Omega}_I$.

Proof:

Let $v = \sum_{i=1}^{I} w_i + \nabla\phi \in \tilde{X}$be any function with $w_i \in H_0(curl, \tilde{\Omega}_i)$ and $\phi \in H^1(\Omega)$.

Let $\phi_i \in H_0^1(\tilde{\Omega}_i)$ solve the elliptic problems

$$\int_{\tilde{\Omega}_i} \nabla\phi_i \cdot \nabla\varphi = \int_{\tilde{\Omega}_i} w_i \cdot \nabla\varphi \quad for\ all\ \varphi \in H_0^1(\tilde{\Omega}_i), \quad 1 \le i \le I.$$

Then we have $v = \sum_{i=1}^{I} v_i + \nabla\psi$ with$v_i = w_i - \nabla\phi_i \in \tilde{X}_i$and $\psi = \phi + \sum_{i=1}^{I}\phi_i \in H^1(\Omega)$. The Poincare type inequality shows that

$$\|v_i\|_{H(curl,\Omega)} \le C\|curl\ v_i\|_{L^2(\tilde{\Omega}_i)} = C\|curl\ w_i\|_{L^2(\tilde{\Omega}_i)} = C\|curl\ v\|_{L^2(\tilde{\Omega}_i)},$$

$$\|\nabla\psi\|_{L^2(\Omega)} \le \|v\|_{L^2(\Omega)} + \sum_{i=1}^{I}\|v_i\|_{L^2(\Omega)} \le C\|v\|_{L^2(\Omega)}$$

,

Where the constant C only depends on $\tilde{\Omega}_i$.

The uniqueness of the decomposition follows from the stability estimate.

2. EDDY CURRENT MODEL FOR NONDESTRUCTIVE EVALUATION

Theorem

The space $\tilde{X}$ admits the decomposition in a direct sum

$$\tilde{X} = \tilde{U} + \sum_{i=1}^{M} U_i, \quad \tilde{U} = \{v \in U : div\ v = 0 \quad in \quad D_1 \cup \dots \cup D_M\}.$$

Proof:

Clearly $\tilde{U} + \sum_{i=1}^{M} U_i \subset \tilde{X}$. The inverse inclusion only requires showing $U \subset \tilde{U} + \sum_{i=1}^{M} U_i$. For any $v \in U$ and any $1 \le i \le M$, let $\phi_i \in H_0^1(D_i)$ solve the elliptic problem

$$\int_{D_i} \nabla\phi_i \cdot \nabla\varphi = \int_{D_i} v \cdot \nabla\varphi \qquad \forall\varphi \in H_0^1(D_i).$$

We extend ϕ_i by zero to the exterior of D_i.Since$\cup_{i=1}^{M}\bar{D}_i \subset D_c$, we have $\tilde{v} = v - \sum_{i=1}^{M}\nabla\phi_i \in \tilde{U}$ and thus $v \in \tilde{U} + \sum_{i=1}^{M} U_i$.To prove the direct sum, we take any $\tilde{v} \in \tilde{U}$ and

$v_i \in U_i, 1 \le i \le M$, satisfying

$$\tilde{v} + \sum_{i=i}^{M} v_i = 0.$$

Then for each $1 \le i \le M$. there exists a$\phi_i \in H^1_{\partial D_i \setminus \bar{S}_i}(D_i)$ such that

$$\bar{v} = 0 \; in \;\; \Omega \setminus (\bar{D}_1 \cup \ldots \cup \bar{D}_M) \;\; and \;\; \tilde{v} = \nabla \phi_i \;\; in \;\; D_i.$$

Since$\bar{v} \in H_0(curl, \Omega)$, the second equality implies that

$$\nabla \phi_i \times n = 0 \;\; in \;\; \partial D_i.$$

This shows $\phi_i \in H^1_0(D_i)$ for each$1 \le i \le M$, and the definition of $\tilde{U}$ implies $\Delta \phi_i = 0$ in D_i, and thus $\phi_i = 0$in D_i.

Therefore, $\tilde{v} \equiv 0$in Ω. So

$$U = \{v \in H_0(curl, \Omega): (v, \nabla\varphi) = 0 \;\; \forall \varphi \in H^1_0(\Omega), \; \varphi = const. in \; \bar{D}_c\}$$

is a direct sum.

Theorem

Assume $J_s \in L^2(\Omega), div\, J_s = 0,$ and $supp(J_s) \cap D_c = \emptyset$.Then the solution of problem $a(\tilde{u}, v) = (J_s, v) \;\; \forall v \in \tilde{X}$satisfies

$$div\,(\tilde{\sigma}\tilde{u}) = 0 \; in \; D_c,$$

$$\tilde{\sigma}\tilde{u} \cdot n = 0 \;\; on \; \partial D_c \cup S_1 \cup \ldots \cup S_M.$$

Proof:

Given that,$J_s \in L^2(\Omega), div\, J_s = 0$,and $supp(J_s) \cap D_c = \emptyset$.

To prove
$$div\,(\tilde{\sigma}\tilde{u}) = 0 \; in \; D_c, \quad \tilde{\sigma}\tilde{u} \cdot n = 0 \; on \; \partial D_c \cup S_1 \cup \ldots \cup S_M.$$

From the equation,

$$U = \{v \in H_0(curl, \Omega): (v, \nabla\varphi) = 0 \; \forall \varphi \in H^1_0(\Omega), \varphi = const. in \; \bar{D}_c\},$$

any$v \in H_0(\mathrm{curl}, \Omega)$ admits an orthogonal decomposition

$$v = v_\perp + \nabla\varphi$$

Where $v_\perp \in U$ and $\varphi \in H^1_0(\Omega)$ satisfying $\varphi = const. in \; \bar{D}_c$.

Since $supp(J_s) \cap D_c = \emptyset$ and $supp(\tilde{\sigma}) = \bar{D}_c$, we have

$$i\omega(\tilde{\sigma}\tilde{u}, \nabla\varphi) + v_0(c\tilde{u}rl\; \tilde{u}, c\tilde{u}rl\; \nabla\varphi) = (J_s, \nabla\varphi)$$

Thanks to $a(\tilde{u}, v) = (J_s, v) \;\; \forall v \in \tilde{X}$ and $v_\perp \in \tilde{X}$, we have

$$i\omega(\tilde{\sigma}\tilde{u}, v_\perp) + v_0(c\tilde{u}rl\; \tilde{u}, c\tilde{u}rl\; \mathrm{v}_\perp) = (J_s, \mathrm{v}_\perp)$$

Adding up the above two equalities yields

$$i\omega(\tilde{\sigma}\tilde{u}, v) + v_0(c\tilde{u}rl\; \tilde{u}, c\tilde{u}rl\; \mathrm{v}) = (J_s, \mathrm{v})$$

$$\forall v \in H_0(curl, \Omega)$$

It implies that $a(\tilde{u}, v) = (J_s, v) \;\; \forall v \in \tilde{X}$ holds for a larger test function space, namely,

$$i\omega(\tilde{\sigma}\tilde{u}, v) + v_0(c\tilde{u}rl\; \tilde{u}, c\tilde{u}rl\; \mathrm{v}) = (J_s, \mathrm{v})$$

$$\forall v \in H_0(curl, \Omega) + \sum_{i=1}^{M} U_i$$

Now taking $v = \nabla\varphi$ for all $\varphi \in H^1_0(\Omega)$ shows that

$$div\,(\tilde{\sigma}\tilde{u}) = 0 \; in \; D_c, \qquad \tilde{\sigma}\tilde{u} \cdot n = 0 \; on \; \partial D_c,$$

and it follows that $[\tilde{\sigma}\tilde{u} \cdot n]_{S_i} = 0$ for $1 \le i \le M$.

Furthermore, since the equation holds for all $v \in U_i$, we also have

$$(\tilde{\sigma}\tilde{u})_{D_i} \cdot n = 0 \; on \; S_i,$$

Where $(\tilde{\sigma}\tilde{u})_{D_i}$ is understood to take the limit of $\tilde{\sigma}\tilde{u}$ from inside D_i.

This means $\tilde{\sigma}\tilde{u} \cdot n = 0 \; on \; S_i$for all $1 \le i \le M$.

3.AN EFFICIENT EDDY CURRENT MODEL FOR NONLINEAR MAXWELL EQUATIONS

Theorem

Let $X_{odd} + \chi \cdot \nabla H^1_0(\Omega)$ is a direct sum in the sense that, for any $\hat{v} \in X_{odd}$ and $v \in H^1_0(\Omega)$,

$$\hat{v} + \chi\nabla v = 0 \qquad if \; and \; only \; if \;\; \hat{v} = 0, \;\; \chi\nabla v = 0. \qquad (1)$$

Moreover, $\tilde{X}$ is a Hilbert space under the inner product and norm

$$(v, w)_{\tilde{X}} = \int_{\tilde{\Omega}_c} v \cdot w + \int_{\Omega} c\tilde{u}rl\; v \cdot c\tilde{u}rl\; w,$$

$$\|v\|_{\tilde{X}} = \sqrt{(v, v)_{\tilde{X}}} \qquad \forall v, w \in \tilde{X}. \qquad (2)$$

Proof:

Suppose that $\hat{v} + \chi\nabla v = 0$ in Ω for $\hat{v} \in X_{odd}$ and $v \in H_0^1(\Omega)$. Then

$\hat{v} = -\nabla v \;\; in \;\; \widetilde{\Omega}_{even}$ and $\hat{v} = 0$ elsewhere.

We know that $v_i = v \,|_{\widetilde{\Omega}_i}$ solves the elliptic problem

$$-\Delta v_i = div\, \hat{v} = 0 \;\; in\, \widetilde{\Omega}_i, \quad \nabla v_i \times n = \hat{v} \times n = 0 \;\; on\, \partial\widetilde{\Omega}_i, (3)$$

for any $1 < i \le I$ and even i. Clearly (3) only has constant solutions so that $\nabla v_i = 0$ in $\widetilde{\Omega}_i$. We have $\chi\nabla v = 0$ and $\hat{v} = 0$ in Ω. Thus (1) is a direct sum.

Next we prove that $\tilde{X}$ is complete. Since $\chi \cdot \nabla H_0^1(\Omega)$ is isomorphic to $\nabla H^1(\widetilde{\Omega}_{even})$, it suffices to prove the completeness of X_{odd}. One equivalent norm on X_{odd} is defined by

$$\|v\|_{X_{odd}} = \left(\int_{\widetilde{\Omega}_{odd}} |v|^2 + \int_{\Omega} |curl\, v|^2\right)^{\frac{1}{2}} \quad \forall v \in X_{odd}.$$

Let $\{v_n\}_{n=1}^{\infty} \subset X_{odd}$ be a Cauchy sequence under the norm $\|\cdot\|_{X_{odd}}$. Then it is also a Cauchy sequence under $\|\cdot\|_{H(curl,\Omega)}$. There exists a $v \in H_0(curl, \Omega)$ such that

$$\lim_{n\to\infty} \|v_n - v\|_{H(curl,\Omega)} = 0,$$

$$(v, \nabla\varphi) = \lim_{n\to\infty}(v_n, \nabla\varphi) = 0 \qquad \forall\varphi \in H_{odd}^1(\Omega).$$

Thus $v \in X_{odd}$ and $\lim_{n\to\infty}\|v_n - v\|_{X_{odd}} = 0$. Then X_{odd} is complete, and so is $\tilde{X}$.

Now let $v = \hat{v} + \chi\nabla \mathrm{v}$ satisfy $\|v\|_{\tilde{X}} = 0$ where $\hat{v} \in X_{odd}$ and $v \in H_0^1(\Omega)$.Then (2) show that

$$\hat{v} = 0 \;\; in \;\; \widetilde{\Omega}_{odd}, \;\; \hat{v} + \nabla v = 0 \;\; in\, \widetilde{\Omega}_{even},$$

$$curl\hat{v} = 0 \;\; in \;\; \Omega.$$

This indicates $\|\hat{v}\|_{X_{odd}} = 0$.We conclude that $\hat{v} = 0$ in Ω and thus $v = 0$ in Ω. Therefore, $\|\cdot\|_{\tilde{X}}$ is a norm on $\tilde{X}$ so that $\tilde{X}$ is a Hilbert space.

Theorem

Let $0 < \sigma_{min} \le \sigma \le \sigma_{max}$ in Ω_c and $\sigma \equiv 0 \;\; in \;\; \Omega_{nc} = \Omega \setminus \overline{\Omega}_c, 0 < v_{min} \le H_i^{'}(B_i) \le v_{max}$ a.e in $\Omega, i = 1,2,3$, and

$J_s \in L^2(0, T; L^2(\Omega))$ and $divJ_s = 0 \;\; in \;\; \Omega$ be satisfied.

Then

$$\int_{\Omega} \tilde{\sigma}\frac{\partial\tilde{u}}{\partial t}\cdot v + \int_{\Omega} \widetilde{H}(c\tilde{u}rl\,\tilde{u})\cdot c\tilde{u}rl\, v = \int_{\Omega} J_s \cdot v$$

$\forall v \in \tilde{X}(1)$

has at most one solution.

Proof:

To prove

$$\int_{\Omega} \tilde{\sigma}\frac{\partial\tilde{u}}{\partial t}\cdot v + \int_{\Omega} \widetilde{H}(c\tilde{u}rl\,\tilde{u})\cdot c\tilde{u}rl\, v = \int_{\Omega} J_s \cdot v$$

$\forall v \in \tilde{X}$ has at most one solution.

Suppose $\tilde{u}_1$ and $\tilde{u}_2$ are two solutions of equation (1).

Then

$$\int_{\Omega} \tilde{\sigma}\frac{\partial}{\partial t}(\tilde{u}_1 - \tilde{u}_2)\cdot v + \int_{\Omega}\{\widetilde{H}(c\tilde{u}rl\,\tilde{u}_1) - \widetilde{H}(c\tilde{u}rl\,\tilde{u}_2)\}\cdot c\tilde{u}rl\, v = 0 \quad \forall v \in \tilde{X}.$$

It means that,

for almost every $t \in (0, T]$ and all $v \in L^2(0, t; \tilde{X})$,

$$\int_0^t \int_{\Omega} \tilde{\sigma}\frac{\partial}{\partial t}(\tilde{u}_1 - \tilde{u}_2)\cdot v + \int_0^t \int_{\Omega}\{\widetilde{H}(c\tilde{u}rl\,\tilde{u}_1) - \widetilde{H}(c\tilde{u}rl\,\tilde{u}_2)\}\cdot c\tilde{u}rl\, v = 0.$$

Taking $v = \tilde{u}_1 - \tilde{u}_2$, the above equality shows that

$$\int_0^t \frac{d}{dt}\int_{\Omega} \frac{\tilde{\sigma}}{2}|\tilde{u}_1-\tilde{u}_2|^2 + \int_0^t\int_{\Omega} \{\tilde{H}(c\tilde{u}rl\,\tilde{u}_1) - \tilde{H}(c\tilde{u}rl\,\tilde{u}_2)\}\cdot c\tilde{u}rl\,(\tilde{u}_1-\tilde{u}_2) = 0$$

From initial conditions $\tilde{u}_1(\cdot\ ,0) = \tilde{u}_2(\cdot\ ,0) = 0$ in $\tilde{\Omega}_c$, we find that

$$\int_0^t \frac{d}{dt}\int_{\Omega} \frac{\tilde{\sigma}}{2}|\tilde{u}_1-\tilde{u}_2|^2 = \frac{1}{2}\int_{\Omega} \tilde{\sigma}|\tilde{u}_1(t)-\tilde{u}_2(t)|^2.$$

And the strict monotonicity of $\tilde{H}$ shows that

$$\int_{\Omega} \{\tilde{H}(c\tilde{u}rl\,\tilde{u}_1) - \tilde{H}(c\tilde{u}rl\,\tilde{u}_2)\}\cdot c\tilde{u}rl\,(\tilde{u}_1-\tilde{u}_2) \geq v_{min}\,\|c\tilde{u}rl\,(\tilde{u}_1-\tilde{u}_2)\|^2_{L^2(\Omega)}.$$

It follows that, for almost every $\in (0,T]$,

$$\frac{1}{2}\int_{\Omega} \tilde{\sigma}\,|\tilde{u}_1(t)-\tilde{u}_2(t)|^2 + v_{min}\int_0^t \|c\tilde{u}rl\,(\tilde{u}_1-\tilde{u}_2)\|^2_{L^2(\Omega)} \leq 0.$$

This shows $\|\tilde{u}_1-\tilde{u}_2\|_{L^2(0,T;\tilde{X})} = 0$.We have $\tilde{u}_1 = \tilde{u}_2$.

4. CONCLUSION

In this paper, we established some theorems about an H-ψ formulation for the three dimensional eddy current problem and we have some ideas about this topic.

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