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Classification of Number Theory

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***Abstract**: Number theory is the science of studying the properties of numbers. It has a wide range of applications. Due to advance of computers it becomes possible to solve many problems of numbers. According to objectives and research methods, we can divide number theory into four classes: elementary number theory, analytic number theory, algebraic number theory and geometric number theory.*

***Key Words**: Analytic, Prime, Twin, Algebraic, Geometric*

1. **Elementary Number Theory:** Elementary or classical number theory is the basic theory for studying divisibility, congruence, Diophantine equations, congruence equations etc., mainly by means of the four fundamental rules (addition, subtraction, multiplication and division). It is based on calculations and method of proof involves only basic deduction.
2. **Analytic Number Theory:** In number theory the subject which can be studied by using analytic method i.e. Method of the theory of functions and mathematical analysis is called the analytic number theory. Euler was the first person who uses analytic methods to study number theory. So he is originator of analytic number theory. Analytic methods push the research of number theory to a new height. Some problems can not be solved without analytic methods, while some others can be solved more simply by analytic methods. The innovation was led by G.L. Dirichlet (1805-1859) and F.B. Riemann (1826-1859), who developed the number theory by studying the following several problems.

   (2.1) Prime Number Theorem: A prime number can be divided evenly only by 1 or itself. And it must be a whole number greater than 1. Example: 2, 3,5, 7, 11 etc. The distribution of primes among the positive integers is irregular. For example, it was discovered by Gauss that there is no prime in the 26379th hundred numbers, i.e. there is no prime between 2637801 and 2637900, but there are seventeen primes in the 27050th hundred numbers.

   The prime counting function is the function $\pi(n)$ giving the number of primes less than or equal to a given number n [1]. This notation was introduced by number theorist Edmund Landau in 1909 and has now become standard. For example, there are no primes ≤ 1, so $\pi(1) = 0$. There is a single prime (2) ≤ 2 , so $\pi(2) = 1$. There are two primes (2 and 3) ≤ 3 , so $\pi(3) = 2$ and $\pi(10) = 4,\ \pi(100) = 25,\ \pi(150) = 35$ and so on. However there is no explicit expression for it.

In 1846, Bertrand conjectured that when $2a > 7$ there was at least a prime between a and $2a - 2$.
P.A. Chebyshev (1821-1894) proved this conjecture in 1852.Desloves conjectured that there were at least two primes between n^2 and $(n+1)^2$.
Prime number theorem (PNT) describes the asymptotic distribution of the prime numbers among the positive integers.
The famous prime number theorem states that for large n, $\pi(n)$ approaches $\frac{n}{\log n}$ i.e. $\pi(n) \sim \frac{n}{logn}$

Or $\lim_{n\to\infty} \frac{\pi(n)}{n/\log n} = 1$

In 1792, when only 15 years old, Gauss proposed that $\pi(n) \sim \frac{n}{\log n}$

Gauss later refined his estimate to
$Li(n) = \int_2^n \frac{dx}{\log x}$

is the logarithmic integral. Gauss did not publish this result, which he first mentioned in an 1849 letter to Encke. It was subsequently posthumously published in 1863 [2].
Legendre (1808) suggested that for large n-

$$\pi(n) \sim \frac{n}{logn + B}$$

with $B = -1.08366$ (where B is sometimes called Legendre's constant)
Li(n) has the asymptotic series about infinity of
$Li(n) \sim \sum_{k=0}^{\infty} \frac{k!\, n}{(\log n)^{k+1}}$

$$\sim \frac{n}{\log n} + \frac{n}{(logn)^2} + \frac{2n}{(\log n)^3} + \cdots \ldots \ldots \ldots \ldots \ldots$$

and taking the first three terms has been shown to be a better estimate than $\frac{n}{\log n}$ alone [3].

Table of $\pi(n), n/\log n\ \ and\ Li(n)$ [4]

The table compares exact values of π(n) to the two approximations n / log *n* and Li(N). The last column, *n* / π(n), is the average prime gap below n.

n	*π*(*n*)	*π*(n) − *n*/ log *n*	$\frac{\pi(n)}{n/\log n}$	Li(n) − *π*(*n*)	*n* /*π*(*n*)
10	4	−0.3	0.921	2.2	2.500
10^2	25	3.3	1.151	5.1	4.000
10^3	168	23	1.161	10	5.952
10^4	1,229	143	1.132	17	8.137
10^5	9,592	906	1.104	38	10.425
10^6	78,498	6,116	1.084	130	12.740
10^7	664,579	44,158	1.071	339	15.047
10^8	5,761,455	332,774	1.061	754	17.357
10^9	50,847,534	2,592,592	1.054	1,701	19.667
10^{10}	455,052,511	20,758,029	1.048	3,104	21.975
10^{11}	4,118,054,813	169,923,159	1.043	11,588	24.283
10^{12}	37,607,912,018	1,416,705,193	1.039	38,263	26.590
10^{13}	346,065,536,839	11,992,858,452	1.034	108,971	28.896
10^{14}	3,204,941,750,802	102,838,308,636	1.033	314,890	31.202

10^{15}	29,844,570,422,669	891,604,962,452	1.031	1,052,619	33.507
10^{16}	279,238,341,033,925	7,804,289,844,393	1.029	3,214,632	35.812
10^{17}	2,623,557,157,654,233	68,883,734,693,281	1.027	7,956,589	38.116
10^{18}	24,739,954,287,740,860	612,483,070,893,536	1.025	21,949,555	40.420
10^{19}	234,057,667,276,344,607	5,481,624,169,369,960	1.024	99,877,775	42.725
10^{20}	2,220,819,602,560,918,840	49,347,193,044,659,701	1.023	222,744,644	45.028
10^{21}	21,127,269,486,018,731,928	446,579,871,578,168,707	1.022	597,394,254	47.332
10^{22}	201,467,286,689,315,906,290	4,060,704,006,019,620,994	1.021	1,932,355,208	49.636
10^{23}	1,925,320,391,606,803,968,923	37,083,513,766,578,631,309	1.020	7,250,186,216	51.939
10^{24}	18,435,599,767,349,200,867,866	339,996,354,713,708,049,069	1.019	17,146,907,278	54.243
10^{25}	176,846,309,399,143,769,411,680	3,128,516,637,843,038,351,228	1.018	55,160,980,939	56.546

J.Hadmard (1865-1963) and C.de la Vallee Poussian (1866-1962) independently and almost simultaneously succeeded in proving the theorem .

In 1948, Norwegian mathematician A. Selberg and Hungarian mathematician P. Erdos gave an elementary proof of the theorem. Their proof depends on an inequality.

About the lower and upper bounds of $\pi(n)$, P.A.Chebyshev theorem states that if $n \geq 2$ then

$.79 \frac{n}{\log n} \leq \pi(n) \leq 1.38 \frac{n}{\log n}$ [5]

The zeta function: Euler (1748) had studied the series

$\zeta(s) = 1 + 1 2 s + 1 3 s + 1 4 s + \ldots,$

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots \ldots \ldots \ldots$$

(convergent for s > 1). Euler factored this over the primes,

$$\zeta(s) = \left(1 + \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{8^s} + \cdots \ldots\right)\left(1 + \frac{1}{3^s} + \frac{1}{6^s} + \cdots \ldots\right)\left(1 + \frac{1}{3^s} + \frac{1}{9^s} + \cdots \ldots\right) \ldots \ldots \ldots \ldots$$

and used geometric series,

$$\sum_{k=0}^{\infty} \left(\frac{1}{p^s}\right)^k = \left(1 - \frac{1}{p^s}\right)^{-1}$$

To conclude $\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$

From $\lim_{s \to 1} \zeta(s) = \infty$, we can deduce that $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots = \infty.$

By use of zeta function and the method of contradiction Euler proved that there are infinitely many primes.

(2.2) Dirichlet's Theorem:

In 1837, Dirichlet prove that for any two positive coprime integers a and b i.e. if $(a, b) = 1$ there are infinitely many primes of the form $a + xb$, where x is a non-negative integer. The numbers of the form $a + xb$ form an arithmetic progression. This result is known as Dirichlet's theorem on the existence of primes in a given arithmetic progression.

and Dirichlet's theorem states that this sequence contains infinitely many prime numbers. The theorem extends Euclid's theorem that there are infinitely many prime numbers. Stronger forms of Dirichlet's theorem state that for any such arithmetic progression, the sum of the reciprocals of the prime numbers in the progression is divergent; this generalizes Euler's result that the sum of the inverses of all primes is divergent. Dirichlet's theorem greatly helped the development of the analytic number theory.

(2.3) Goldbach's Problem

In 1742 , C.Goldbach (1690-1764) conjectured that every odd number greater than $\geq$ 9 is expressible as the sum of three old primes, and every even number $\geq$6 is expressible as the sum of two odd primes.

For example $11 = 3 + 3 + 5,\ 13 = 3 + 5 + 5,\ 15 = 5 + 5 + 5,\ 17 = 5 + 5 + 7,$

$19 = 5 + 7 + 7,\quad 21 = 7 + 7 + 7, 23 = 5 + 7 + 11,\ 25 = 5 + 7 + 13,$

$27 = 7 + 7 + 13$ and so on

& $6 = 3 + 3,\quad 8 = 3 + 5,\quad 10 = 5 + 5,\quad 12 = 5 + 7,$

$14 = 7 + 7,\ 16 = 5 + 11,\ 18 = 5 + 13,\ 20 = 7 + 13,\ 22 = 11 + 11,\ 24 = 11 + 13$ and so on.

Sometimes the first conjecture is called the old Goldbach's conjecture or "weak" Goldbach conjecture. , and the second the even Goldbach's conjecture. Goldbach's original conjecture (sometimes called the "ternary" Goldbach conjecture), written in a June 7, 1742 letter to Euler, states "at least it seems that every number that is greater than 2 is the sum of three primes" [6]. here Goldbach considered the number 1 to be a prime, a convention that is no longer followed. Vinogradov proved that every sufficiently large odd number is the sum of three primes [7, 8], and Estermann proved that almost all even numbers are the sums of two primes [9].

(2.4) The Twin prime problem A pair of successive odd prime numbers that differ by 2 are called twin primes.For example, (3,5); (5,7); (11,13);(17, 19); (29,31); (41,43) ...

It is not known whether the set of twin prime numbers ends or not. The twin prime conjecture states that there is infinite pair of twin primes. As Goldbach's conjecture it is unsolved problem, and has become one of the central problems in prime number theory. During his research , Jing- Run Chen showed that there exist infinitely many primes p such that p+2 is a prime or is the product of at most two primes.

Twin primes are pair of peimes whose difference is 2. There are also pairs of primes whose difference is 4, such as (3,7); (9,11); (13,17); (19,23); (37,41); (43,47); (67,71);……….

Alphonse de Polignac (1826–1863) was a French mathematician. In 1849, the year he was admitted to Polytechnique, he made what's known as Polignac's conjecture: For every positive integer *k*, there are infinitely many prime gaps of size 2*k*.

The case *k* = 1 is the twin prime conjecture.

Every even number is the difference of two consecutive primes in infinitely many ways [10]. If true, taking the difference 2, this conjecture implies that there are infinitely many twin primes [11].The conjecture has never been proven true or refuted.

(2.5) Waring's Problem: On the additive number theory Diophantus had a famous conjecture which states that a positive integer can be expressed as a sum of four squares. For example

$1 = 1^2 + 0^2 + 0^2 + 0^2,\quad 2 = 1^2 + 1^2 + 0^2 + 0^2$

$3 = 1^2 + 1^2 + 1^2 + 0^2,\ 4 = 1^2 + 1^2 + 1^2 + 1^2$

$5 = 2^2 + 1^2 + 0^2 + 0^2,\ 6 = 2^2 + 1^2 + 1^2 + 0^2, 7 = 2^2 + 1^2 + 1^2 + 1^2,$

$8 = 2^2 + 2^2 + 0^2 + 0^2, 9 = 2^2 + 2^2 + 1^2 + 0^2,\ 10 = 3^2 + 1^2 + 0^2 + 0^2$

$11 = 3^2 + 1^2 + 1^2 + 0^2, 12 = 3^2 + 1^2 + 1^2 + 1^2$

The first person who published his proof was J.L.Lagrange (1736-1813), in 1770.Three years later , Euler gave a simpler proof, when he was 66 years old and at that time he was blind.

In 1770, E.Waring (1737-1798) conjectured that every positive integer can be expressed as a sum of four squares, a sum of nine cubes, and a sum of nineteen number quartics.

In 1909,D.Hilbert (1862-1943) showed the following more general conclusion.

For every integer $k \geq 2$, there exists a positive integer $r = r(k)$

Such that every positive integer N can be expressed as $N = {x_1}^k + {x_2}^k + \cdots \ldots \ldots \ldots \ldots + {x_r}^k$

Where $x_i \geq 0$ are integers. Let $g(k)$ be the minimal integer r with the above property. Now $g(2)$ and $g(3)$ have been found, namely $g(2) = 4$ and $g(3) = 9$.

Jing-Rur Chen obtained $g(5) = 37$ in 1964 and $g(4) \leq 27$ in 1974. H.E. Thomas showed that $19 \leq g(4) \leq 22$.

3. Algebraic Number Theory

α **is** called the algebraic number theory if it is a root of an algebraic equation

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots .. + a_{n-1}x + a_n$$

, where $a_i{}'s$ are integers. α **is called an algebraic number of degree n if f(x)** is an irreducible polynomial of degree n, and an algebraic integer if $a_0 = 1$. Non algebraic integers are called transcendental numbers. The sum, difference, product and quotient (zero can not be the denominator) of two algebraic numbers are also algebraic numbers. The sum, difference and product of two algebraic integers are also algebraic integers. But the quotient of two algebraic integers may not be an algebraic integer. A field whose elements are algebraic numbers is called an algebraic number field. Let Q denote the rational field , and $Q(\alpha)$ the algebraic extension field of Q by adding an algebraic number α of degree n. $Q(\alpha)$ is called an algebraic field over Q of degree n obviously, the set of all algebraic integers contained in $Q(\alpha)$ forms a ring, called an algebraic integral ring. Algebraic number theory deals with the algebraic integers in $Q(\alpha)$, or the algebraic integral ring.

After Dedekind and other mathematicians systematically established the concept of ideal, module and Dedekind domain, a new science "algebraic number theory" was founded. One of the most important developments of algebraic number theory is the creation of ideal numbers. E.E.Kummer (1810-1893) andd J.W..R.Dedekind (11831-1916) introduced the "ideal" number. All ideal numbers can be classified into ideal number classes. Study of ideal numbers and the computation of ideal number classes are important problems of algebraic number theory.

4. Geometric Number Theory

In a plane orthogonal coordinate system, a point (x, y) is called an integral point or lattice if the coordinates x and y are integers. A very famous and difficult unsolved problem in number theory is Gauss's integral point problem: How many integral points are there inside the circle with centre at the origin and radius$r$? Denote by $A(r)$ the number of integral points inside the circle and on the boundary. Gauss's problem becomes to seek the relationship between $A(r)$ and r. Since the area of circle with radius r is πr^2, it is conjectured that $A(r) = [\pi r^2]$.

Sierpinski (1882-1969) showed that

$$A(r) = \pi r^2 + 0(r^{2/3} \log r)$$

In 1942 Lu- Keng Hua obtained an improvement of the above result, namely

$$A(r) = \pi r^2 + 0[r^{13/20} (\log r)^{9/8}]$$

In 1963 Jing-Run Chen obtained

$$A(r) = \pi r^2 + 0(r^{3/4+8}$$

H. Minkowski (1864-1909) Showed that there must exist a non zero integral point inside (including the boundary) symmetric convex whose volume is greater than 2^n.

In general, dealing with problems of number theory by geometric methods such as the above integral point method is called geometric number theory.

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