Observer based Adaptive Synchronization of Chaotic Systems With Parametric Uncertainties Using Generalized Hamiltonian Approach

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Abstract: This paper uses and modifies the Generalized Hamiltonian approach in synchronization of two chaotic systems in transmitter-receiver configuration. The transmitter is assumed to have known parameters while the receiver in this case has unknown parameters. The receiver is a nonlinear observer for the transmitter belonging to the class of chaotic systems which is represented in Generalized Hamiltonian form. The transmitter and receiver are synchronized using Generalized Hamiltonian Approach. In most practical scenarios, parameters are mostly unknown and may change from time to time. Therefore, how to effectively synchronize chaotic systems with unknown parameters is an important problem for theoretical research and practical application. In this paper, the systems with destabilizing vector field or dissipation are considered. Based on Lyapunov theory, parameter update equations are formed which ultimately makes the error in states to reduce to zero. Numerical simulations are used to study the model and their synchronization is explained based on these terms.

1. Introduction
The field of synchronization of chaos has grown many folds since its advent in 1990. Chaos has long-term unpredictable behavior which is associated with system’s sensitivity to initial conditions. A chaotic system can have an attractor or pattern in state space but determining where on the attractor the system is at a future time, given its position in the past is a problem that becomes hard as time passes. One way to show this is to run two, identical chaotic systems side by side, starting both at close, but not exactly equal initial conditions.

The noise like behavior of chaotic systems suggested early on that such behavior might be useful in secure and private communications. The huge interest in the topic of synchronization arises from the possibilities of encoding messages using the chaotic signals generated as states of a chaotic system called the “transmitter”. The random nature of the carrier signal, modulated by the masked message signal, makes it difficult for the intruder in the channel to attempt the decoding of the message from the intercepted signal.

Suggestions to use chaos in robotics or biological implants have surfaced in recent times. If several parts of a system are required to act together, although chaotically, synchronization of chaos is needed. In general, it is desired to achieve such synchronization using a minimal number of signals between the synchronous parts. Using a single signal and passing it among them would be optimal.

The first thing to be highlighted is that there is a great difference in the process leading to synchronized states, depending upon the type of coupling configuration. The two main cases for chaotic coupling are: unidirectional coupling and bidirectional coupling. In former, one subsystem evolves freely and drives the evolution of the other, while in latter, both subsystems are coupled with each other and interaction is allowed both ways. For the case of coupled chaotic elements, many different synchronization states have been studied in the past like complete or identical synchronization [2-4], phase synchronization [5,6], lag synchronization [7], generalized synchronization [8, 9], intermittent lag synchronization [7,10], and imperfect phase synchronization [11], to name a few. The first to be discovered was Complete Synchronization (CS) and is the simplest form of synchronization in chaotic systems. It consists of a perfect hooking of the chaotic trajectories of two systems which is achieved by using a coupling signal, in such a way that they remain in sync with each other in the course of time. In laboratory experiments and natural systems [12-22], the phenomenon of synchronization has been studied. In [23,27], the authors have studied the synchronization of two chaotic systems by the generalized Hamiltonian approach. Furthermore, the method is extended to the time-delay Chua’s oscillator [24].
In this article, the synchronization issue of chaotic systems, from the perspective of Generalized Hamiltonian systems, with receiver having unknown parameters, including the non conservative terms is addressed. A large number of chaotic systems can be placed in such a Generalized Hamiltonian canonical form, from where the reconstructability of the state vector, from a defined output signal, may be assessed from the observability or detectability of a pair of constant matrices. Most of the existing methods can synchronize two identical or different chaotic systems with known parameters. However, in practical engineering situations, parameters are mostly unknown and may change from time to time. Therefore, how to effectively synchronize chaotic systems with unknown parameters is an important problem for theoretical research and practical application. Among the aforementioned methods, adaptive control is an effective one for achieving the anti-synchronization of chaotic systems with unknown parameters. On the basis of the Lyapunov stability theory, we design a new adaptive synchronization controller with a novel parameter update law. The Generalized Hamiltonian structure of most known chaotic systems allows us to decide clearly on the nature of synchronizing signal. To analyze the efficacy of the approach, numerical simulations on Rossler chaotic system are presented at the end.

### 2. Nonlinear Observer Design for a Class of Systems in Generalized Hamiltonian Form

In this section, problem formulation for designing nonlinear observer from systems in Generalized Hamiltonian form is presented. Consider a smooth system as given below:

\[
\dot{x} = f(x, t), \quad x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n
\]  

(2.1)

where \( f \in \mathbb{R}^n \) is smooth vector field defined in \( \mathbb{R}^n \times \mathbb{R} \) and \( x \) is \( n \)-dimensional state vector.

The equation (2.1) may be rewritten in the Generalized Hamiltonian system [27] as follows:

\[
\dot{x} = J(x) \frac{\partial H}{\partial x} + S(x) \frac{\partial H}{\partial x} + F(x, t) \]  

(2.2)

where \( H(x) \) denotes a smooth energy function and is globally positive definite in \( \mathbb{R}^n \), and we assume that the column gradient vector \( \frac{\partial H}{\partial x} \) of \( H(x) \) exists everywhere, if the form of quadratic energy function is \( H(x) = \frac{1}{2} x^T M x \) (\( M \) is a constant symmetric positive definite matrix), \( \frac{\partial H}{\partial x} = Mx \). Also, \( J(x) + J(x)^T = 0 \), \( S(x) = S(x)^T \). The vector field \( J(x) \frac{\partial H}{\partial x} \) exhibits the conservative part of the system and it is also referred to as the workless part and \( S(x) \) depicts the working part of the system. \( F(x, t) \) is a locally destabilizing vector field. According to the form of \( H(x) \) and the different expression of \( J(x), S(x), F(x) \), the form of the Generalized Hamiltonian System (2.2) is not unique.

For the case of Generalized Hamiltonian Systems with unknown parameters, we consider a special class of systems with linear output map, \( y \), given as

\[
\begin{align*}
\dot{x} &= J(x) \frac{\partial H}{\partial x} + (I + S) \frac{\partial H}{\partial x} + F(x, t) \phi \\
y &= C \frac{\partial H}{\partial x} 
\end{align*}
\]  

(2.3)

where \( J(x) + J(x)^T = 0 \), \( I \) is constant skew symmetric matrix, \( S \) is constant symmetric matrix. Here, \( y \) is system output along with a constant matrix \( C \). The equation (2.3) is generally referred to as “Transmitter”. \( F(x, t) \) is the destabilizing vector field and \( \phi \) is the unknown parameter vector.

Now, we need to construct a state estimator for \( x \) as well as \( y \). Let the estimate of \( x \) be denoted as \( \hat{x} \) and estimate of \( y \) be denoted as \( \hat{y} \). We consider the Hamiltonian function \( H(\xi) \) to be a particularization of \( H \) in terms of \( \xi \). Here \( \xi \) is calculated in terms of the estimated state \( \epsilon \). The gradient vector \( \frac{\partial H(\xi)}{\partial \epsilon} \) is of the form of \( M \xi \) with \( M \) being a constant symmetric positive definite matrix. \( F(\xi, t) \) represents the destabilizing field with \( \phi \) as unknown parameters vector.

A dynamic non linear state observer is constructed for equation (2.3) and is written as:

\[
\begin{align*}
\dot{\epsilon} &= J(x) \frac{\partial H}{\partial x} + (I + S) \frac{\partial H}{\partial x} + F(x, t) \phi + K(y - \eta) \\
\eta &= C \frac{\partial H}{\partial x} \\
\dot{\phi} &= -F(x, t) \left( \frac{\partial H}{\partial x} \right)^T 
\end{align*}
\]  

(2.4)

where \( K \) is constant vector, known as Observer Gain. \( \phi \) is the estimation of parameters for the given observer. \( \phi \) is the parameter update equation. The equation (2.4) may be called as “Receiver”.

The main focus of this paper is to study the synchronization of the transmitter (2.3) and the receiver (2.4).

Initially the observer will not be an exact match to the states, therefore, there will be estimation errors which we need to reduce, preferably to zero. The state estimation error is defined as \( \epsilon(\xi) = x(\xi) - \epsilon(\xi) \) and the output estimation error is defined as \( \epsilon_y = y(t) - \eta(t) \). Then, these are governed by,

\[
\begin{align*}
\dot{\epsilon} &= J(x) \frac{\partial H}{\partial x} + (I + S) \frac{\partial H}{\partial x} + F(x, t) (\phi - \phi) + K(y - \eta) \\
\dot{\eta} &= C \frac{\partial H}{\partial x} \\
\dot{\phi} &= -F(x, t) \left( \frac{\partial H}{\partial x} \right)^T 
\end{align*}
\]  

(2.5)

where the vector \( \frac{\partial H}{\partial x} \), stands for the gradient vector of the modified energy function, \( \frac{\partial H}{\partial x} = \frac{\partial H}{\partial x} \frac{\partial H}{\partial \epsilon} = M \xi \).
Proof:- Let the Lyapunov functional be
\[ V(e(t), \tilde{\theta}) = H(e(t)) + \frac{1}{2} \tilde{\theta}^2. \]
The derivative of this functional with respect to time \( t \) is following:
\[ \dot{V} = \left( \frac{\partial H}{\partial e} \right)^T \dot{e} + \tilde{\theta} \]
\[ \dot{V} = \left( \frac{\partial H}{\partial e} \right)^T (J(x) + I + S-KC) \frac{\partial H}{\partial e} + F(x,t)\tilde{\theta} + \tilde{\theta} \]
as \( \tilde{\theta} - \theta \) which implies \( \dot{\tilde{\theta}} = \dot{\theta} \).

Substituting,
\[ \dot{V} = \left( \frac{\partial H}{\partial e} \right)^T J(x) \frac{\partial H}{\partial e} + \left( \frac{\partial H}{\partial e} \right)^T F(x,t)\tilde{\theta} + \tilde{\theta} \]
or
\[ \dot{V} = \left( \frac{\partial H}{\partial e} \right)^T S \frac{1}{2} (KC + C^TK^T) \frac{\partial H}{\partial e} + \left( \frac{\partial H}{\partial e} \right)^T F(x,t)\tilde{\theta} + \tilde{\theta} \]
or
\[ \dot{\tilde{\theta}} = - \left( \frac{\partial H}{\partial e} \right)^T F(x,t) \]
which is the required parameter update equation.

In [1], the authors point out that the transmitter (2.3) synchronizes with the receiver (2.4), if
\[ ||x(t)-e(t)|| \to 0, \]
no matter what initial conditions \( x(0) \) and \( e(0) \) have. The state estimation error \( e(t) = x(t)-e(t) \) represents the synchronization error. So we will study the system (2.5) for the synchronization. In the following, two theorems about (2.5) give the conditions under which their synchronization happens. Let \( W = I + S \).

**Definition 2.1** [23] Given a pair of constant matrices \((C, A)\) with size \( m \times n \) and \( n \times n \) respectively, then, the pair is said to be detectable if the matrix
\[ C - S W A \]
has full rank \( n \) for all values of \( s \) in the open right half of complex plane. The system is said to be observable if the above matrix is full rank for all values of \( s \) in the complex plane.

**Theorem 2.1** [23] The state \( x(t) \) of the nonlinear system (2.3) can be globally exponentially asymptotically estimated by the state \( e(t) \) of the nonlinear observer (2.4), if the pair of matrices \((C, W)\) or the pair \((C, S)\), is either observable or, at least, detectable.

An observability condition on either of the pairs \((C, W)\) or \((C, S)\), is clearly a sufficient but not necessary condition for asymptotic state reconstruction. A necessary and sufficient condition for global asymptotic stability to zero of the estimation error is given by the following theorem.

**Theorem 2.2** [23] The state \( x(t) \) of the nonlinear system (2.3) can be globally exponentially asymptotically estimated by the state \( e(t) \) of the nonlinear observer (2.4), if and only if there exists a constant matrix \( K \) such that the symmetric matrix
\[ [W-KC] + [W-KC]^T = [S-KC] + [S-KC]^T = 2 \left[ S \frac{1}{2} (KC + C^TK^T) \right] \]
is negative definite.

### 3. Numerical Applications
To show the efficacy of the procedure highlighted in section 2, numerical simulations are performed on various chaotic systems in this section.

#### 3.1 Lorenz System with unknown parameters
Consider now, Lorenz chaotic system with the following system equations [26]:

\[
\begin{align*}
\dot{x}_1 &= \sigma(x_2-x_1) \\
\dot{x}_2 &= rx_1-x_2-x_1x_3 \\
\dot{x}_3 &= -bx_1 + x_2x_3
\end{align*}
\]

The Hamiltonian energy function is taken as
\[ H(x) = \frac{1}{2} x_1^2 + x_2^2 + x_3^2 \]

After taking (3.1.2) as Hamiltonian energy function, we obtain:
\[ J = \begin{bmatrix}
\frac{1}{2} \sigma & 0 & 0 \\
-x_1 & \frac{1}{2} \sigma & -1 \\
0 & 0 & -b
\end{bmatrix} \quad I + S = \begin{bmatrix}
0 & \frac{1}{2} \sigma & -1/2 \\
\frac{1}{2} \sigma & -1 & 0 \\
0 & 0 & -b
\end{bmatrix}
\]

F\((x,t) = \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
\]

Now, we write the Generalized Hamiltonian canonical form for (3.1.1) as
\[\begin{align*}
\dot{x}_1 &= 0 \\
\dot{x}_2 &= \frac{1}{2} \sigma x_1 - x_1 \\
\dot{x}_3 &= \frac{1}{2} \sigma x_1 - x_1
\end{align*}\]

We choose \( y = [x_1] \) such that \( C = [1 \ 0 \ 0] \) and let
\[ K = \begin{bmatrix}
K_1 \\
K_2 \\
K_3
\end{bmatrix}
\]
The pair of matrices \((C, S)\) already constitute a detectable, but not observable pair. The addition to \( S \) of matrix \( I \) doesn’t improve the lack of observability. The unstable nature of observable eigenvalues of \( S \) requires the introduction of damping through the output error injection map and proceed to place the eigenvalues of observable part of the dissipative structure of the reconstruction error in suitable (asymptotically) stable locations in the complex plane. This results in following non linear receiver dynamics:
The synchronization error, corresponding to this receiver is

$$e_1 = -e_2 + (r-x_1)e_3 + K_2 e_2$$

The parameter update equation is

$$\dot{r} = -e_2 x_1$$

In Figure 3.1.1, the synchronization of chaotic system in (3.1.3) and its observer in (3.1.4) is presented with the following parameter values

$$\sigma = 10, b = 2.667, r = 28, K_1 = -1, K_2 = 1, K_3 = 0$$

The Fig 3.1.1(a) shows the 3-D Lorenz chaotic system (3.1.3) in general with parameters as specified above. Fig. 3.1.1(b) shows that the errors are reducing to zero as the time progresses, confirming the synchronization between system states and observer states.

3.2 Forced Pendulum System with unknown parameters

Considering a forced pendulum chaotic system as given below:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a x_2 - b \sin(x_1) + p \cos(\omega t)$$

and the Hamiltonian energy function as

$$H(x) = \frac{1}{2} [x_1^2 + x_2^2 + x_3^2]$$

we obtain:

$$J(x) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad I = \begin{bmatrix} 0 & 1/a \\ 1/a & 0 \end{bmatrix},$$

$$S = \begin{bmatrix} 0 & 1/a \\ 1/a & 0 \end{bmatrix},$$

$$F(x, t) = \begin{bmatrix} -b \sin(x_1) + p \cos(\omega t) \\ \end{bmatrix}$$

Now, we write the Generalized Hamiltonian canonical form for (3.1.1) as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1/a \\ 1/a & 0 \end{bmatrix} \frac{\partial H}{\partial x} + \begin{bmatrix} 0 & 1/a \\ 1/a & 0 \end{bmatrix} \frac{\partial H}{\partial \dot{x}} +$$

$$\begin{bmatrix} -b \sin(x_1) + p \cos(\omega t) \end{bmatrix}$$

We choose $y = [x_1]$. The pair of matrices $(C, S)$ already constitute a detectable, but not observable pair. The addition to $S$ of matrix $I$ doesn’t improve the lack of observability. The unstable nature of observable eigenvalues of $S$ requires the introduction of damping through the output error injection map.
and proceed to place the eigenvalues of observable part of the dissipative structure of the reconstruction error in suitable (asymptotically) stable locations in the complex plane. Assuming the uncertain parameter to be $\theta$ results in following non linear receiver dynamics:

$$\begin{bmatrix}
\dot{\xi}_1 \\
\dot{\xi}_2 \\
\dot{\xi}_3
\end{bmatrix} = \begin{bmatrix}
0 & \frac{1-a}{2} & \frac{1+a}{2} \\
\frac{1-a}{2} & 0 & \frac{1+a}{2} \\
0 & \frac{1-a}{2} & -p(\cos(\omega t)) + \frac{K_1}{K_2} (Y - \varepsilon)
\end{bmatrix} \begin{bmatrix}
\xi_1 \\
\xi_2 \\
\xi_3
\end{bmatrix}$$

The synchronization error, corresponding to this receiver, is

$$\begin{bmatrix}
\dot{e}_1 \\
\dot{e}_2 \\
\dot{e}_3
\end{bmatrix} = \begin{bmatrix}
e_1 + K_1 e_1 \\
(-a e_1 + (b-b) \sin(x_1) + K_2 e_2_1)
\end{bmatrix}$$

The parameter update equations then become

$$\dot{\theta} = -e_2 \sin(x_1)$$

In Figure 3.2.1, the synchronization of chaotic system in (3.1.1) and its observer in (3.2.2) is presented with the following parameter values $a = 0.2, b = 1, p = 1.5, \omega = 0.4, K_2 = -(1-a)$

Fig. 3.2.1 shows the convergence of errors to zero as the time progresses, confirming the synchronization between system states and observer states.

3. Conclusion

In this paper, the issue of synchronization of chaotic systems with parametric uncertainties from the perspective of Generalized Hamiltonian systems including dissipation and destabilizing terms is presented. A few advantages of this approach are that it makes the design of non linear observer simple and allows to decide clearly on the nature of output signal to be transmitted. This is achieved using a simple observability or detectability test. Some chaotic systems were analyzed from this new perspective and their synchronization was confirmed with simulated results.

4. References


