Imperial Journal of Interdisciplinary Research (IJIR)
Vol-2, Issue-5, 2016
ISSN: 2454-1362, http://www.onlinejournal.in

A Study on Central and Peripheral Support in Graphs

A. Anto Kinsley[1] & J. Joan Princiya[2]
[1]Associate Professor, [2]Research Scholar
Department of Mathematics, St. Xavier's College (Autonomous), Palayamkottai - 627 002

Abstract: *We consider finite, connected, undirected simple graphs with n vertices. In this paper, we study different structures of vertices in a graph. In particular, we define and determine the support vertices of central and peripheral. The relations between eccentricity, radius, and diameter of such graphs are also investigate***..**

Introduction

By a graph $G = (V, E)$, we mean a finite, undirected, connected graph without loop or multiple edges. For a graph G, let $V(G)$ and $E(G)$ denote is vertex and edge set respectively and n and m. In this section, some definitions and important results on center and peripheral vertices [3] are studied.

Definition1.1

For a connected graph G and a pair u, v of vertices of G, the *distance* $d(u, v)$ between u and v is the length of a shortest u-v path in G.

Definition 1.2

For a vertex v in a connected graph G, the *eccentricity* $e(v)$ of v is the distance between v and a vertex farthest from v in G. Thus $e(v)=max\{ d(u, v)/u\epsilon V\}$.

Definition 1.3

The minimum eccentricity among the vertices of G is its *radius* and the maximum eccentricity is its *diameter*, which are denoted by $rad(G)$ and $diam(G)$ respectively. That is, $r(G)=min\{ e(v)/v\epsilon V(G)\}$ and $diam(G)=max\{ e(v)/v\epsilon V(G)\}$.

Definition 1.4

A vertex v in G is a *central vertex* if $e(v)= rad(G)$ and a subgraph induced by the central vertices of G is the *center* of G and it is denoted by $cen(G)$.

Definition 1.5

If every vertex of G is a central vertex then $cen(G)=G$ and G is called *self-centered*.

1. Re-formation of a graph with central vertices

Definition 2.1

A vertex v in a connected graph G is called a *peripheral vertex* if $e(v)=diam(G)$. The subgraph of G induced by its peripheral vertices is the *periphery* of G and it is denoted by $per(G)$.

Consider the following example,

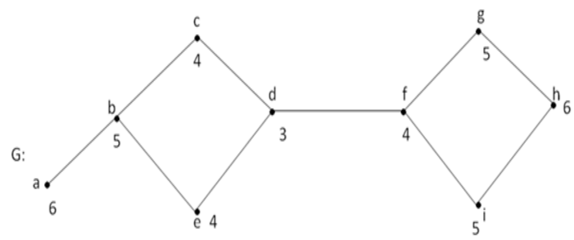

Figure 1: A graph with eccentricities

Here, Per(G)={a, h}. The following figure represents the structure of a graph G involving the sets center, periphery and support vertices.

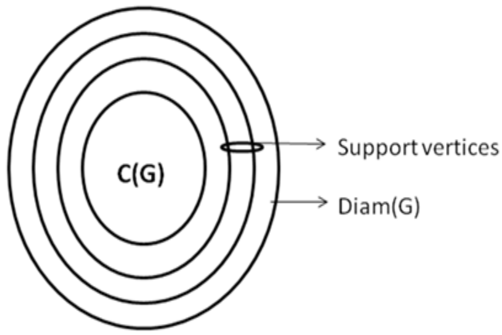

Figure 2: Structure of a graph G

A vertex x with eccentricity $e(x)$ such that $r(G) \leq e(x)$ $d(G)$ is called a *support vertex* and is denoted by $S(G)$. The center of G is given by $C(G)=\{ x\epsilon V / e(x)= r(G)\}$. Consider all vertices in center and keep them in a circle. Draw outer layer to $C(G)$ and locate the support vertices in the layer whose eccentricity is equal to $r(G)+1$.Draw another outer layer to the above said layer for the support vertices having eccentricity as $r(G)+2$. Similarly draw corresponding circles for support vertices. Finally the outer most circle should be drawn for peripheral vertices. The following are the immediate results regard support vertices.

(i). When $d(G)= r(G)$, no support vertices appear. Here $C(G)=per(G)$.

(ii). When $d(G)- r(G)=1$, no support vertices appear. Here $C(G)=V-per(G)$.

(iii). When $d(G)- r(G)=2$, there exist a layer between central and peripheral layers, the vertices having eccentricity $r(G)+1$ are support vertices and so $S(G)=V- per(G)- C(G)$.

(iv). When $d(G)- r(G)=3$, there exist a layers having support vertices with eccentricity $r(G)+1$ and $r(G)+2$ (i.e., $diam(G)-1$). In this case some vertices are near to central vertices and some vertices are near to peripheral vertices. We call these types of vertices as central support vertices (CSV) and peripheral support vertices (PSV) respectively.

(v). When $d(G)- r(G)=4$, there exist three layers having support vertices with eccentricity $r(G)+1$, $r(G)+2$ and $r(G)+3$ (i.e., $diam(G)-1$). In this case we have three types of vertices.

Among them, two cases are same as the previous case and an extra layer is visible between the above two layers. This layer of vertices will not support the central and peripheral vertices.

(vi). When $d(G)- r(G)>4$, there exist more than three layers. Among the layers, the layers mentioned in the case(v) are visible and more number of layers visible which have non-support vertices.

Hence in general, we can have five different structures of vertices such as, we can have five different structures of vertices such as

(i). Central vertices
(ii). Central support vertices
(iii). Non-support vertices
(iv). Peripheral support vertices
(v). Peripheral vertices

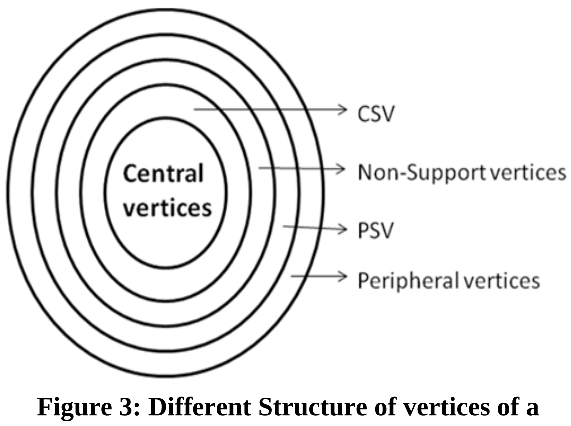

Figure 3: Different Structure of vertices of a graph G

Let w be the central support vertex. Let x be an eccentric vertex of a central vertex v.
Then $d(x, v) = r(G)$. Let y be a neighboring vertex on the eccentric path from v to x. Then join w with y. Then w will have the eccentricity $r(G)$.Then $w \epsilon C(G) \rightarrow (1)$.

If there are n number of central support vertices, then join them with the neighbors as like as the previous procedure. Finally we will have a enlarged central vertex set. The cardinality of the center of new graph will be $|C(G)|+n$.

Proof of (1): Let v be the central vertex. Then $e(v) = r(G)$ and so $d(x, v) = r(G)$ and so $d(y, v) = r(G)-1$, by the assumption of y. It implies $d(w, v) = d(w, y)+ d(y, v)=1+ r(G)-1$ and so $d(w, v)= r(G)$.

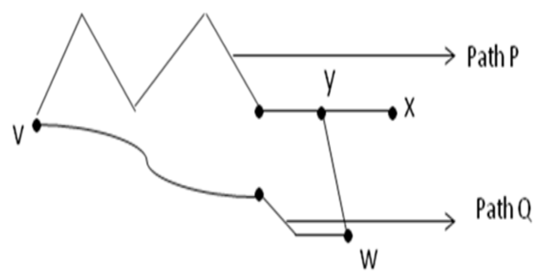

Figure 4: Central and Support vertices of G

In the figure
$d(v, x) = e(v)$ by P
$d(x, y) =1$ in P.

Let G' be the new graph having enlarged center. Now consider the non-support central vertices these vertices will keep their status in G' as in G. The graphs in case (ii) have no peripheral vertices. In case (iii), G' has no support vertex. In case (iv) G' has one layer between central and peripheral vertices.

Theorem 2.2

For every pair of central vertex u and central support vertex x in a connected graph, $e(x)- e(u)=1$.

Proof

Let u be a central vertex and x be a central support vertex. Let v be a vertex that is farthest from u. i.e. $d(u, v) =e(u)$.Similarly, $d(x, v) = e(x)$.
That is, $e(x)=d(x, v)= d(x, u)+ d(u, v)=1+ e(u)$. Hence, $e(x)- e(u)=1$. ■

Theorem 2.3

For every pair of central vertex u and central support vertex x in a connected graph, $d(x, v)- d(u, v)=1$.

Proof

Let u be a central vertex and x be a central support vertex. Let v be a vertex that is farthest from u. i.e., $d(u, v)$ is a shortest u-v path in G. In the same way, *d*(*x* ,*v*) is a *x*-*v* geodesic in *G*. Hence, *d*(*x* ,*v*)= *d*(*x* ,*u*)+*d*(*u* ,*v*)=1+*d*(*u* ,*v*).which implies *d*(*x* ,*v*)- *d*(*u* ,*v*)=1. ■

Theorem 2.4

Every graph is the periphery of some graph.

Proof

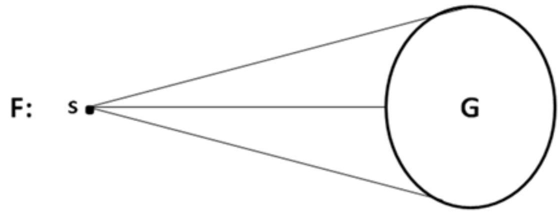

Figure 5: A graph G with newly added vertex s

Let *G* be a connected graph. First, add a new vertex *s* to *G* and join it to every vertex of G. Consider this new graph as *F*. Since *e*(*s*) =1 and *e*(*x*) = 2 for every vertex $x \in G$, it follows that *V*(*G*) is the set of peripheral vertices of *F* and so *per*(*F*)=*G*. ■

Note: The above mentioned graph *F* has no support vertex.

Definition 2.5

A connected graph *G* is an *eccentric graph* if every vertex of *G* is an eccentric vertex.

Example:

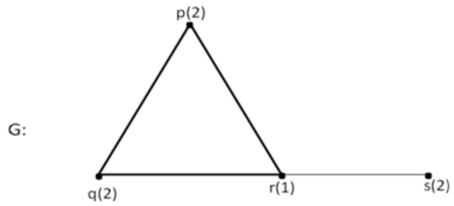

Figure 6: Eccentric graph G

Definition 2.6

Let *G* be a connected graph. The *eccentric sub graph* Ecc (G) of *G* is the subgraph of *G* induced by the set of eccentric vertices of *G*.

Example:

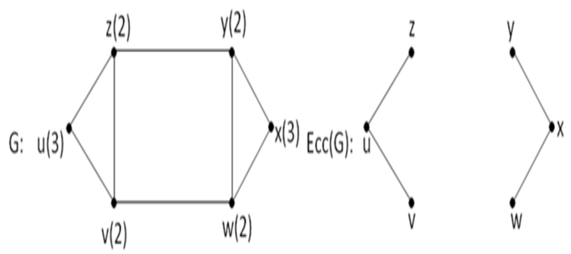

Figure 7: A graph G with eccentric subgraph

Definition 2.7

A vertex *v* in a connected graph G is a *boundary vertex* of a vertex *u* if *d*(*u* ,*w*)≤ *d*(*u* ,*v*).for each neighbor *w* of *v*; while a vertex *v* is a boundary vertex of the graph G if *v* is a boundary vertex of some vertex of G.

Example:

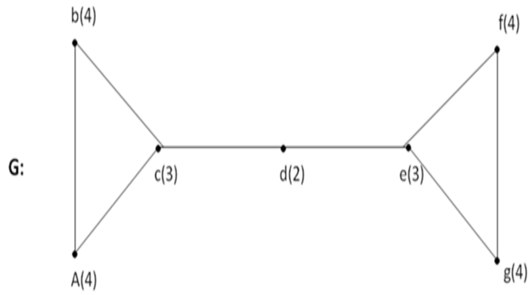

Figure 8: Boundary vertices of G

We note the following, In a connected graph, every peripheral vertex is an eccentric vertex. Every eccentric vertex is a boundary vertex. Every boundary vertex is an end vertex but every boundary vertex need not be an eccentric vertex.

Definition 2.8

A vertex *v* in a graph *G* is called a *complete vertex* (or *an extreme* or *simplicial vertex*) if the sub graph of *G* induced by the neighbors of *v* is complete. In particular, every end vertex is complete**.**

Theorem 2.9

Every peripheral vertex, which is not a boundary vertex, is not complete.

Proof

Let *G* be a connected graph. Let *v* be a peripheral vertex in *G*, which is not a boundary vertex. Suppose *v* is a complete vertex in *G* and let *u* be a vertex distinct from *v*. Also let $u = v_0, v_1, \ldots, v_k = v$ be a *u*-*v* geodesic and let *w* be a neighbor of *v*. if $w = v_{k-1}$, then *d*(*u* ,*w*)< *d*(*u* ,*v*). so we may assume that $w \neq v_{k-1}$. since *v* is complete, $wv_{k-1} \epsilon E(G)$ and $u = v_0, v_1, \ldots, v_{k-1} = w$ is a *u*-*w* path in *G*. which implies *d*(*u* ,*w*)≤ *d*(*u* ,*v*). Hence *v* is a boundary vertex of *G*, which is a contradiction. Hence *v* is not a complete vertex. ■

Definition 2.10

In a connected graph *G*, let *x* and *z* be two distinct vertices in G. A vertex *y* distinct from *x* and *z* is said to lie *between x* and *z* if *d*(*x, z*) ≤ *d*(*x* ,*y*)+ *d*(*y* ,*z*) .

Definition 2.11

A vertex *v* is an *interior vertex* of *G* if for every vertex *u* distinct from *v*, there exists a vertex *w* such that *v* lies between *u* and *w*. The interior Int(*G*) of *G* is the subgraph of *G* induced by interior vertices.

Example:

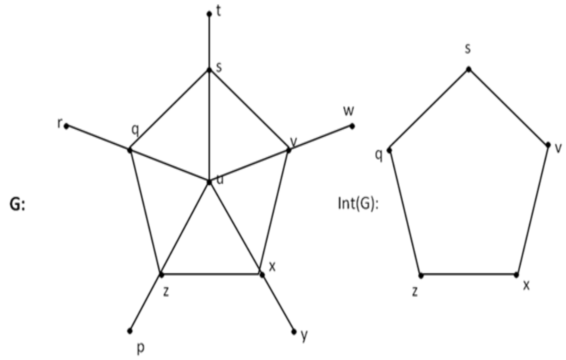

Figure 9: A graph G with interior vertices

Theorem 2.12

Every central vertex of a connected graph G of order $n > 2$ is not an end vertex.

Proof

Let u be a central vertex. Suppose that u is an end vertex. Therefore $e(u)= rad(G)$, which is minimum of all eccentricities. Let v be the eccentric vertex of u. Then $e(u) = d(v,u)= rad(G)$. Let w be a neighbor of u on the eccentric path of u then $d(w,v)= d(v,u)-1$, That is, $d(w,v) = rad(G) -1$. Let x be the eccentric vertex of w. Then $d(w,x)= e(w) \geq rad(G)$. Hence $d(u, x) = d(u, w) + d(w, x) = 1+e(w) \geq 1 + rad(G)$.Thus $d(u, x) \geq 1 + rad(G)$. Hence $d(u, x) \geq 1 + rad(G)$ and so $d(u, x) > rad(G) = e(u)$. It contradicts the fact that u is a central vertex and it implies that u is not an end vertex. ■

References

[1] Anto Kinsley, A., Somasundaram, S., "Computation of Center and Median of Graphs: An Algorithmic Approach", IJST/ vol.1/ISSUE 3/ September, 2013.

[2] Buckley, F., and Harary, F., "Distance in Graphs", Addison-Wesley Publishing Company, New York, 1990.

[3] Buckley, F., Miller, Z., and Slater, P.J.," On graphs containing a given graph as center", Journal Of Graph Theory,5(4): 427-434,1981

[4] Gary Chartrand and Ping Zhang, "Introduction to Graph Theory", Tata McGraw-Hill, New Delhi 2006.

[5] Harary, F., "Graph Theory", Addison-Wesley, 1969.

[6] Mahesh C. Prajapati., "Distance in Graph Theory and its application", IJAET /Vol.II/ ISSUE IV/ October- December, 2011/147-150.

.