Abstract—This paper presents a new method of reduced order observer for linear descriptor systems. A discretization technique for descriptor form linear systems has been discussed and an exact discrete-time model has been developed. The model is illustrated for descriptor matrix of non-singular and singular form. Numerical example strengthens the performance of the proposed observer.

Keywords—Functional observer, Descriptor system, Linear system, exact discrete-time model, reduced order observer.

I. INTRODUCTION

State-space analysis used in an extensive variety of systems for various purposes as modeling, analysis, and design of real-time practical systems, faces few limitations because of the algebraic constraint between the states. This results in overall system complexity. For example consider a mass-spring damper system whose order is reduced by assuming the mass equal to zero. Such a singularly perturbed system has attracted a large number of studies [1,2] and the so-called descriptor-system expression is found to be convenient when dealing with such cases as multi-timescale systems, localization of parameter variations, and design of controllers with variable orders.

For designing any system, it is necessary to have some knowledge about states of the system to give feedback. But sometimes in many applications, it becomes very expensive or becomes impossible or unnecessary to calculate all of these state variables. For such systems or such cases observers are used. Observer for any system is another system used to estimate the states of that system. It is actually a mathematical realization of any system in which input output information is used to trace the output towards the true state values of the given system.

Successful design or analysis of dynamic systems proceeds from a suitably derived mathematical model. Modern control theory is mostly based on state space representation. Due to increase of computers for control purpose, a significant amount of research has been carried out in the field of discrete-time systems. Discrete-time controller design use the discretized representation of the continuous-time system [4,5]. The discretized system is composed of a zero-order hold (ZOH), the continuous-time system and a sampler in series. The state feedback controller is most effective in controlling the system performance, because, the state vector contains all the essential information [1].

The paper is organized as follows: Section 1 represents the Introduction. In section 2 two exact discrete-time models presented for descriptor systems for two different cases - one when the descriptor matrix is non-singular and the other when it is singular. The derivations of these models are given in Section 3, followed by an example for each case in Section 4 with simulation results. In the last section, the paper has been concluded.

II. DISCRETIZATION OF DESCRIPTOR SYSTEM

Discretization in this method is done based on the system which have a zero order hold element at the input side. In other words it can be understand as the use of this method is limited to the systems for designing controllers which should have a sample and hold element at its input, as when a physical process is controlled by computer via a DA converter (digital to analog). Zero order hold means that the input signal is held constant during the time-step or sampling interval [3].

\begin{align*}
E\dot{x}(t) &= Ax(t) + Bu(t) \\
\dot{x}(t) &= E^{-1}Ax(t) + E^{-1}Bu(t) \\
y(t) &= Cx(t)
\end{align*}

The input continuous system (1) and (2) changes at discrete sampling intervals. Hence the discrete time model of the system.

\begin{align*}
x((k+1)T) &= Gx(kT) + Hu(kT) \\
y(kT) &= Cx(kT) + Du(kT)
\end{align*}

Multiply \( e^{-E^{-1}At} \) both sides in eq. (2)

\begin{align*}
e^{-E^{-1}At}(\dot{x}(t) - E^{-1}Ax(t)) &= e^{-E^{-1}At}E^{-1}Bu(t) \\
\frac{d}{dt}[e^{-E^{-1}At}x(t)] &= e^{-E^{-1}At}E^{-1}Bu(t)
\end{align*}

Integrate both side from \((k)T\) to \((k+1)T\)
\[
\int_{t_0}^{t} \frac{d}{dt} [e^{-EAt} x(t)] dt = \int_{t_0}^{t} e^{-EAt} Bu(t) dt \tag{7}
\]

\[
e^{-EAt} x(t) = e^{-EAt} x(t_0) + \int_{t_0}^{t} e^{-EAt} Bu(\tau) d\tau \tag{8}
\]

Multiply \( e^{-EAt} \) both sides, we get

\[
e^{-EAt} \left( e^{-EAt} x(t) \right) = \int_{t_0}^{t} e^{-EAt} Bu(\tau) d\tau \tag{9}
\]

\[
x(t) = e^{-EAt(t-t_0)} + \int_{t_0}^{t} e^{-EAt(\tau-t_0)} Bu(\tau) d\tau
\]

We will now determine the value of the matrices and . It will turn out that while they are constant for a particular sampling interval, they depend on the value of the sampling interval, so for that reason it have written them as \( \lambda \) and \( \lambda \) in (3) above. We start by using the solution of (3) to calculate the values of the state at times \( kT \) and \((k+1)T\). These are

\[
x((k+1)T) = e^{-EAT((k+1)T)} x(0) + e^{-EAT((k+1)T)} \left( \int_{0}^{T} e^{-EAt} Bu(\tau) d\tau \right) \tag{10}
\]

\[
x(kT) = e^{-EAT(kT)} x(0) - e^{-EAT(kT)} \left( \int_{0}^{T} e^{-EAt} Bu(\tau) d\tau \right) \tag{11}
\]

We want to write \( x(k+1) \) in terms of \( x(kT) \) so we multiply all terms of (11) by \( e^{\lambda T} \) and solve \( e^{-EAT(k+1)T} x(0) \) obtaining

\[
e^{-EAT(k+1)T} x(0) = e^{-EAT(kT)} x(kT) - e^{-EAT(k+1)T} \left( \int_{0}^{T} e^{-EAt} Bu(\tau) d\tau \right) \tag{12}
\]

Substituting \( e^{-EAT(k+1)T} x(0) \) for in equation (10) we obtain

\[
x((k+1)T) = e^{-EAT} x(kT) + e^{-EAT(k+1)T} \left( \int_{0}^{T} e^{-EAt} Bu(\tau) d\tau - \int_{0}^{T} e^{-EAt} Bu(\tau) d\tau \right) \tag{13}
\]

Which, by linearity of integration, is equivalent to

\[
x((k+1)T) = e^{-EAT} x(kT) + e^{-EAT((k+1)T)} \left( \int_{0}^{T} e^{-EAt} Bu(\tau) d\tau \right) \tag{14}
\]

Next, we notice that within the interval from \( kT \) to \((k+1)T\), \( u(t) = u(kT) \) is constant, as is the matrix \( B \), so we can take them out of integral to obtain.

\[
x((k+1)T) = e^{-EAT} x(kT) + e^{-EAT((k+1)T)} \left( \int_{kT}^{(k+1)T} e^{-EAt} Bu(\tau) d\tau \right) \tag{15}
\]

We can take the \( e^{-EAT((k+1)T)} \) inside the integral to obtain

\[
x((k+1)T) = e^{-EAT} x(kT) - \int_{0}^{T} e^{-EAT} Bu(\tau) d\tau \tag{16}
\]

Or

\[
x((k+1)T) = e^{-EAT} x(kT) - \int_{0}^{T} e^{-EAT} Bu(\tau) d\tau \tag{17}
\]

We see that in (15) we have written the state update equation exactly in the form of (3) where

\[
G = e^{-EAT} \tag{18}
\]

\[
H = \int_{0}^{T} e^{-EAT} Bu(\tau) d\tau \tag{19}
\]

Solution of (3) is thus

\[
x(kT) = (G)_{k} x(0) + \sum_{j=0}^{k-1} (G)^{k-j-1} H u(jT) \quad K=1,2,3\ldots \tag{20}
\]

And we can see that at the sampling instants \( kT \) this has exactly the same value as is obtained using (1). Specifically,

\[
(G)^{k} = (e^{-EAT})^{k} = e^{-EAT} \tag{21}
\]

And since input \( u(t) \) is constant on sampling intervals,
\[
e^{-E^*AT} \int_0^{(kT)} e^{-E^*A\tau} Bu(\tau) d\tau = \\
\sum_{j=0}^{k-1} e^{-E^*A(k-j-1)T} A^{-1}(e^{-E^*AT} - I)Bu(jT) = \\
\sum_{j=0}^{k-1} (G)^{k-j} Hu(jT); \tag{21}
\]

### III. DESIGN APPROACH

Consider continuous-time descriptor system of LTI (linear time-invariance)

\[
Ex(t) = Ax(t) + Bu(t) \tag{22}
\]

\[
y(t) = Cx(t) \tag{23}
\]

The discretization of continuous-time system (22) at sampling period “T” will be

\[
Ex(k+1) = \tau \tilde{A}x(k) + \tilde{B}u(k) \tag{24}
\]

Above equation can be represented by

\[
Ex(k+1) = A_x x(k) + B_d u(k) \tag{25}
\]

Hence the discrete-time representation will be

\[
Ex(k+1) = A_x x(k) + B_d u(k) \tag{26}
\]

\[
y(k) = Cx(k) \tag{27}
\]

Assumption the matrix “E” is square and inverse of matrix E exist. Here the dimensions of matrices are \( R^{nxn} , A \in R^{nxn} , B \in R^{nxm} , \) and \( C \in R^{pxn} \)

The aim of the design method is to present a procedure to estimate function of order \((n-p)\).

\[
z(k) = Fx(k) \tag{28}
\]

Where \( z(k) \in \mathbb{R}^{xn} \) and \( F \in \mathbb{R}^{(n-p)xn} \).

In this section, we present an algorithm design to estimate a function of the form. The proposed observer must to asymptotically estimate the desired function. Here the proposed observed will be in the form.

\[
\hat{z}(k) = w(k) + E_d y(k) \tag{29}
\]

\[
w(k+1) = Nw(k) + Jy(k) + Hu(k) \tag{30}
\]

where \( w(k) \in \mathbb{R}^{(n-p)x(n-p)} \) in an observer state vector and \( \hat{z}(k) \) denotes the estimate of \( z(k) \) the objective is to determine matrices \( N \in \mathbb{R}^{(n-p)x(n-p)} \), \( J \in \mathbb{R}^{(n-p)xn} \), \( H \in \mathbb{R}^{(n-p)xp} \) such that \( \hat{z}(k) \) converges to \( z(k) \) and \( E_d \in \mathbb{R}^{(n-p)xn} \)

**Theorem 1** for the proposed observer (29), (30), the estimate \( \hat{z}(k) \) will converge asymptotically to \( z(k) \) if and only if the following hold

- \( N \) is stable matrix ;
- \( LE + E_d C - F = 0 \)
- \( NLE = JC - LA_d = 0 \)
- \( H - LB_d = 0 \)

**Proof**

If \( \hat{z}(k) \) estimate \( z(k) \), then \( w(k) \) will asymptotically converge \( Lx(k) \). Define the following error dynamics

\[
e(k) = w(k) - LEx(k) \tag{31}
\]

\[
e(k) = \hat{z}(k) - z(k) \tag{32}
\]

Error dynamics, from (31) and using equations (25),(26) and (29)

\[
\epsilon(k + 1) = w(k + 1) - LEx(k + 1) \tag{33}
\]

\[
= Nw(k) + Jy(k) + Hu(k) - L\{A_x x(k) + B_d u(k)\}
\]

\[
= Nw(k) + JCx(k) + Hu(k) - LA_d x(k) - L B_d u(k)
\]

\[
= N\{c(k) + LE(k)\} + JCx(k) +
\]

\[
Hu(k) - LA_d x(k) - L B_d u(k)
\]

\[
\epsilon(k + 1) = N\epsilon(k) +
\]

\[
(NLE + JC - LA_d) x(k) + (H - LB_d) u(k)
\]

From \( e(k) = \hat{z}(k) - z(k) \)

Using (29) in above

\[
e(k) = w(k) + E_d y(k) - z(k)
\]

\[
e(k) = w(k) + E_d Cx(k) - Fx(k)
\]

Substitute the value of \( w(k) \) from equation (31) in above

\[
e(k) = LE(k) + \epsilon(k) + E_d Cx(k) - Fx(k)
\]

\[
= \epsilon(k) + (LE + E_d C - F)x(k)
\]

Which proves theorem.

### IV. NUMERICAL EXAMPLE

Continuous-time model:
Discrete-time form with sampling period 0.1 sec,
\[
\begin{bmatrix}
1 & 0 \\
0 & 10
\end{bmatrix}
\begin{bmatrix}
\hat{x}_1(t) \\
\hat{x}_2(t)
\end{bmatrix}
= 
\begin{bmatrix}
0 & 1 \\
-5 & -3
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix}
+ 
\begin{bmatrix}
0 \\
1
\end{bmatrix}
u(t)
\]
\[
y(t) = 
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix}
\]

Function to be estimated is
\[
z(k) = [0.5, -1]
\begin{bmatrix}
x_1(k) \\
x_2(k)
\end{bmatrix}
\]
\[
A_d = 
\begin{bmatrix}
-0.215 & 0.795 \\
-3.96 & -4.54
\end{bmatrix},
B_d = 
\begin{bmatrix}
0.04 \\
0.793
\end{bmatrix}
\text{ and } F = [0.5, -1]
\]

Now it is very clear that, rank of 
\[
\begin{bmatrix}
1 & 0 \\
0.5 & -1
\end{bmatrix}
\]
=2, hence a reduced order observer can be designed. The model considered above had two states (n=2), one input (m=1) and one output (p=1) Hence the design of reduced order of the order (n-p)= 1 is possible. Solving following equations
\[
LE + E_dC - F = 0 	ext{ and } \]
\[
NLE + JC - LA_d = 0
\]
simultaneously results in : 
\[N = 0.2, J = 1.45 \text{ and } =2.356.\] Hence, the reduced order observer will be
\[
\hat{z}(k) = w(k) + 2.356y(k)w(k + 1)
\]
\[
= 0.2w(k) + 1.45y(k) + 2.356u(k)
\]

In order to validate the above design method, we need to find the response of estimate and it should be compared with true state. The simulations can be carried for initial condition \(x(0) = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \). Fig 1 shows the comparison of \(z(k)\) and \(\hat{z}(k)\).

V. CONCLUSION

From fig 1, it is quite apparent that the estimate of function \(z(k)\) is an approximate. Hence, the design of observer is very much effective in estimation of function. Fig 2 shows the response of the system when function estimated by observer is feedback to the system, i.e., the estimated function \(\hat{z}(k)\) is utilized to for control purpose. The simulation results observe that proposed design method of observer will asymptotically estimate the output.

References


