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On Generalized $\mathcal{BR}$ −Recurrent Affinely Connected Space

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Abstract. *In this paper, we introduced the generalized $\mathcal{BR}-$recurrent Finsler space, i.e. characterized by the following condition*

$$\mathcal{B}_m R^i_{jkh} = \lambda_m R^i_{jkh} + \mu_m(\delta^i_j g_{kh} - \delta^i_k g_{jh}) \ , \quad R^i_{jkh} \neq 0 \ ,$$

which is an affinely connected space, we called it a generalized $\mathcal{BR}-$recurrent affinely connected space, where $\mathcal{B}_m$ is Berwald's covariant differential operator with respect to x^m, λ_m and μ_m are known asrecurrence vectors.
The purpose of the present paper to develop the above space by study the recurrence vectors field and to obtain the condition for some tensors to be recurrent in the generalized $\mathcal{BR}-$recurrent affinely connected space. Also to obtaindifferent theorems for some tensors satisfy in above space. Various identities are established in our space.

Keywords: *Generalized $\mathcal{BR}-$ recurrent affinely connected space, Generalized $\mathcal{B}-$recurrent of Berwald curvature tensor and Generalized $\mathcal{B}-$recurrent of Weyl's curvature tensor.*

1.Introduction

H.D. Pande and B. Singh [10] discussed the recurrence property in an affinely connected space. N.S.H. Hussian [14] introduced and disassedK^h-recurrent affinely connected space. M.A.A. Ali [12], F.Y.A. Qasem and M.A.A. Ali [8] introduced and studied K^h- birecurrent affinely connected space. A.M.A. Hanballa [2], F.Y.A. Qasem and A.M.A. Hanballa [7] introduced and studied K^h-generalized birecurrent affinely connected space. A.N.A. Al−awaidhani [3] disassed C^h-trirecurrent affinely connected space. A.M.A. Al−qashbari [1] introducted and discussed generalized H^h, R^h and K^h-generalized recurrent affinely connected spaces. F.Y.A. Qasem and W.H.A. Hadi [9] discussed generalized $\mathcal{BR}-$birecurrent affinely connected space.

Let F_n be an n – dimensional Finsler space equipped with the metric function F(x,y) satisfying the request conditions [11] .

The vector y_i is defined by

(1.1) $\quad y_i = g_{ij}(x,y)y^j$.

The two sets of quantities g_{ij} and its associative g^{ij}, which are components of a metric tensor connected by

(1.2) $\quad g_{ij}\, g^{ik} = \delta^k_j = \begin{cases} 1 \ \ if \ \ j = k \ , \\ 0 \ \ if \ \ j \neq k \ . \end{cases}$

In view of (1.1) and (1.2), we have

(1.3) $\quad \delta^i_k\, y_i = y_k$,

(1.4)$\delta^i_k\, y^k = y^i$

and

(1.5) $\quad \delta^i_j\, g_{ir} = g_{jr}$.

The tensor C_{ijk} is defined by

$$C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij}$$

which is positively homogeneous of degree -1 in y^i and symmetric in all its indices and called *(h)hv-torsion tensor* [13] and its associative C^i_{jk} is positively homogeneous of degree -1 in y^i and symmetric in its lower indices and called *(v)hv-torsion tensor*. According to Euler's theorem on homogeneous functions, these tensors satisfy the following:

(1.6) $\quad C_{ijk}\, y^i = C_{kij}\, y^i = C_{jki}\, y^i = 0$

and

(1.7) $\quad C^i_{jk}\, y^k = 0 = C^i_{kj}\, y^k$.

The unit vector l^i in the direction of y^i is given by

(1.8) $\quad l^i \coloneqq \frac{y^i}{F}$.

Berwald's covariant derivative $\mathcal{B}_k T^i_j$ of an arbitrary tensor field T^i_j with respect to x^k is given by

$$\mathcal{B}_k T^i_j \coloneqq \partial_k T^i_j - (\dot{\partial}_r T^i_j) G^r_k + T^r_j G^i_{rk} - T^i_r G^r_{jk} .$$

The processes of Berwald's covariant differentiation and the partial differentiation, for an arbitrary tensor field T^i_j, commute according to

$$(\dot{\partial}_k \mathcal{B}_h - \mathcal{B}_k \dot{\partial}_h) T^i_j = T^r_j G^i_{khr} - T^i_r G^r_{khj} \ .$$

Berwald's covariant derivative of the vector y^i vanish identically, i.e.

(1.9) $\quad \mathcal{B}_k y^i = 0$.

The h(v)- torsion tensor satisfies the relation

(1.10) $\quad H^i_{kh}\, y^k = H^i_h = -H^i_{hk}\, y^k$.

Berwald curvature tensor H^i_{jkh} and the h(v) – torsion tensor H^i_{kh} are skew- symmetric in the lower indices k and h and they are positively homogenous of degree zero and one in y^i, respectively. They are satisfy the following :

(1.11) $H^i_{jkh} y^j = H^i_{kh}$

and

(1.12) $H^i_{ki} = H_k$.

The deviation tensor H^i_h is positively homogeneous of degree two in y^i . In view of Euler's theorem on homogeneous functions we have the following relations

(1.13) $H = \frac{1}{n-1} H^r_r$,

where H is the curvature scalar .

The tensor $H_{jk.h}$ defined by

(1.14) $H_{jk.h} \coloneqq g_{ik} H^i_{jh}$.

2. Generalized $\mathcal{BR}$ −Recurrent Affinely Connected Space

A Finsler space whose connection parameter G^i_{jk} is independent of y^i is called an *affinely connected* or *Berwald's space*. Thus, an affinely connected or Berwald's space is characterized by any one of the equivalent conditions

(2.1) a) $G^i_{jkh} = 0$

and

b) $C_{ijk|h} = 0$.

The connection parameters Γ^{*i}_{kh} of Cartan and G^i_{kh} of Berwald coincide in affinely connected space and there are independent of directional argument [11] , i.e. the conditions

(2.2) a) $\dot{\partial}_j G^r_{kh} = 0$

and

b) $\dot{\partial}_j \Gamma^{*i}_{kh} = 0$

are satisfied .

In affinely connected space, Berwald's covariant derivative of the metric tensor g_{ij} vanishes, i.e.

(2.3) $\mathcal{B}_m\, g_{ij} = 0$.

The generalized $\mathcal{B}$R − recurrent Finsler space which is characterized by the condition [4 - 6]

(2.4) $\mathcal{B}_m R^i_{jkh} = \lambda_m R^i_{jkh} + \mu_m\big(\delta^i_j g_{kh} - \delta^i_k g_{jh}\big)$, $R^i_{jkh} \neq 0$

which briefly denoted by $G(\mathcal{B}\mathrm{R}) - RF_n$, where $\mathcal{B}_m$ is Berwald's covariant differential operator with respect to x^m, λ_m and μ_m are known as *recurrence vectors* and the tensor which satisfies the conditions (2.4) is called *generalized* $\mathcal{B}$ *−recurrent* and briefly denotedby $G\mathcal{B} - R$.

In this paper we shall introduce definition for $G(\mathcal{B}R) - RF_n$ possess the properties of affinely connected space as follows:

Definition 2.1. A Finsler space whose Cartan's third curvature tensor R^i_{jkh} satisfies the condition (2.4) which is an affinely connected space [satisfies the conditions (2.1a), (2.1b), (2.2a) and (2.2b)] will be called a *generalized* $\mathcal{B}R -$ *recurrent affinely connected space* and we shall denote it briefly by $G(\mathcal{B}R) - R -$ *affinely connected space*.

Remark 2.1. It will be sufficient to call the tensor which satisfies the condition of $G(\mathcal{B}R) - R -$ *affinely connected space* as *a generalized* $\mathcal{B} -$ *birecurrent tensor* (briefly $G\mathcal{B} - R$).

Let us consider a $G(\mathcal{B}R) - R -$ affinely connected space.

The equations (2.2) [6] and (2.10) [4]

$$\mathcal{B}_m R_{jlkh} = \lambda_m R_{jlkh} + \mu_m\big(g_{jl}\, g_{kh} - g_{kl}\, g_{jh}\big) + 2R^i_{jkh} y^h \mathcal{B}_h C_{ilm} ,$$

$$\mathcal{B}_m W^i_{jkh} = \lambda_m W^i_{jkh} + \mu_m\big(\delta^i_j g_{kh} - \delta^i_k g_{jh}\big) + \frac{2}{n-1} R^i_h y^h \mathcal{B}_h C_{jkm}$$

and the equation ([4] p.113)

$$\mathcal{B}_m P_{rjkh} = \lambda_m P_{rjkh} + \big(g_{jr}\, g_{kh} - g_{kr}\, g_{jh}\big) - 2\, P^i_{jkh} y^h \mathcal{B}_h C_{irm}$$

become

(2.5) $\mathcal{B}_m R_{jlkh} = \lambda_m R_{jlkh} + \mu_m(\, g_{jl}\, g_{kh} - g_{kl}\, g_{jh}\,)$,

(2.6) $\mathcal{B}_m W^i_{jkh} = \lambda_m W^i_{jkh} + \mu_m\big(\delta^i_j g_{kh} - \delta^i_k g_{jh}\big)$

and

(2.7) $\mathcal{B}_m P_{jrkh} = \lambda_m P_{jrkh} + \mu_m\big(g_{jr}\, g_{kh} - g_{kr} g_{jh}\big)$,

respectively .

Thus, we conclude

Theorem 2.1. *In* $G(\mathcal{B}R) - R -$ *affinely connected space, the associate curvature tensor* R_{jrkh} *of Cartan's third curvature tensor* R^i_{jkh}, *Weyl's projective curvature tensor* W^i_{jkh} *and the associate curvature tensor* P_{jrkh} *of Cartan's second curvature tensor* P^i_{jkh} *are* $G\mathcal{B} - R$.

Now, if $\dot{\partial}_j \lambda_m = 0$ and $\dot{\partial}_j \mu_m = 0$, then the eequation (22) [5]

$$\begin{aligned}\mathcal{B}_m H^i_{jkh} + H^r_{kh} G^i_{mjr} - H^i_{rh} G^r_{mjk} - H^i_{kr} G^r_{mjh} &= \big(\dot{\partial}_j \lambda_m\big) H^i_{kh} + \lambda_m H^i_{jkh} \\ &+ \big(\dot{\partial}_j \mu_m\big)\big(y^i g_{kh} - \delta^i_k y_h\big) \\ &+ \mu_m\big(\delta^i_j g_{kh} - \delta^i_k g_{jh}\big) \\ &+ 2y^i \mu_m C_{jkh}\end{aligned}$$

becomes

(2.8) $\mathcal{B}_m H^i_{jkh} = \lambda_m H^i_{jkh} + \mu_m(\delta^i_j g_{kh} - g_{jh}) + 2y^i \mu_m C_{jkh}$.

This shows that

$$\mathcal{B}_m H^i_{jkh} = \lambda_m H^i_{jkh} + \mu_m(\delta^i_j g_{kh} - \delta^i_k g_{jh})$$

if and only if

(2.9) $C_{jkh} = 0$,

since y^i and $\mu_m \neq 0$.

Also, suppose $\dot{\partial}_j \lambda_m = 0$ and $\dot{\partial}_j \mu_m = 0$, the equation (3.13) [6],

$$\mathcal{B}_m H_{jrkh} - 2H^i_{jkh} y^h \mathcal{B}_m C_{irm} + H_{ki.h} G^i_{mjr} - H_{sr.h} G^s_{mjk} -$$

$H_{kr.s}G^s_{mjh} - (\partial_j\lambda_m)H_{kr.h} - \lambda_m H_{jrkh} - \partial_j\mu_m(g_{kh}y_r - y_h g_{kr}) - \mu_m(g_{kh}g_{jr} - g_{jh}g_{kr}) - 2y_r\mu_m C_{ijh} = 0$

becomes

(2.10) $\mathcal{B}_m H_{jrkh} = \lambda_m H_{jrkh} + \mu_m(g_{kh}g_{jr} - g_{jh}g_{kr}) - 2y_r\mu_m C_{ijh}$.

This shows that

(2.11) $\mathcal{B}_m H_{jrkh} = \lambda_m H_{jrkh} + \mu_m(g_{kh}g_{jr} - g_{jh}g_{kr})$

if and only if

(2.12) $C_{ijh} = 0$

since y^i and $\mu_m \neq 0$.

Thus, we conclude

Theorem 2.2. *In $G(\mathcal{B}R) - R-$ affinely connected space, if the directional derivative of covariant vectors field vanish, then Berwald curvature tensor H^i_{jkh} and its associative curvature tensor H_{jrkh} are $G\mathcal{B} - R$ if and only if the (h)hv – torsion tensor C_{ijk} vanishes* .

Again, if $\dot\partial_j\lambda_m = 0$ and $\dot\partial_j\mu_m = 0$, then the equations([5] p.117) $\mathcal{B}_m H_{kh} = \lambda_m H_{kh} + (n-1)\mu_m g_{kh} + H^r_{kh}G^i_{mir} - H^i_{rh}G^r_{mik} - H^i_{kr}G^r_{mih} - (\dot\partial_i\lambda_m)H^i_{kh} - (\dot\partial_i\mu_m)(y^i g_{kh} - \delta^i_k y_h)$

and

$$\mathcal{B}_m(H_{hk} - H_{kh}) = \lambda_m(H_{hk} - H_{kh}) + (n-1)\mu_m g_{kh} + H^r_{kh}G^i_{mir} - H^i_{rh}G^r_{mik} - H^i_{kr}G^r_{mih} - (\dot\partial_i\lambda_m)H^i_{kh} - (\dot\partial_i\mu_m)(y^i g_{kh} - \delta^i_k y_h) = 0,$$

respectively, become

(2.13) $\mathcal{B}_m H_{kh} = \lambda_m H_{kh} + (n-1)\mu_m g_{kh}$

and

(2.14) $\mathcal{B}_m(H_{hk} - H_{kh}) = \lambda_m(H_{hk} - H_{kh}) + (n-1)\mu_m g_{kh}$,

respectively .

Thus, we conclude

Theorem 2.3. *In $G(\mathcal{B}R) - R-$ affinely connected space, if the directional derivative of covariant vectors field vanish, then the H –Ricci tensor H_{kh} and the tensor $(H_{hk} - H_{kh})$ are non – vanishing* .

Also, if $\dot\partial_j\lambda_m = 0$, then the equation (47) [5]

$$\mathcal{B}_m H_{kh} - H_r G^r_{mhk} = (\dot\partial_h\lambda_m)H_k + \lambda_m H_{kh}$$

becomes

(2.15) $\mathcal{B}_m H_{kh} = \lambda_m H_{kh}$.

Thus, we conclude

Theorem 2.4. *In $G(\mathcal{B}R) - R-$ affinely connected space, if the directional derivative of covariant vectors field vanish, then the $H-$ Ricci tensor H_{kh} behaves as a recurrent* .

We know that Berwald curvature tensor satisfies the identity [11]

(2.16) $\mathcal{B}_m H^i_{jkh} + \mathcal{B}_h H^{\ i}_{jmk} + \mathcal{B}_k H^i_{jhm} + H^r_{kh}G^i_{mjr} + H^r_{mk}G^i_{hjr} + H^r_{hm}G^i_{kjr} = 0$.

Let us consider $G(\mathcal{B}R) - R-$ affinely connected space .

Suppose $\dot\partial_j\lambda_m, \dot\partial_j\mu_m = 0$ and by using (22) [5] in (2.16), we get

(2.17) $\lambda_m H^i_{jkh} + \lambda_h H^i_{jmk} + \lambda_k H^i_{jhm} + \mu_m(\delta^i_j g_{kh} - \delta^i_k g_{jh}) + \mu_h(\delta^i_j g_{mk} - \delta^i_m g_{jk}) + \mu_k(\delta^i_j g_{hm} - \delta^i_h g_{jm}) + 2y^i\mu_m C_{jkh} + 2y^i\mu_h C_{jmk} + 2y^i\mu_k C_{jhm} = 0$.

Transvecting (2.17) by y^j , using (1.9), (1.11), (1.4), (1.1) and (1.6), we get

(2.18) $\lambda_m H^i_{kh} + \lambda_h H^i_{mk} + \lambda_k H^i_{hm} + \mu_m(y^i g_{kh} - \delta^i_k y_h) + \mu_h(y^i g_{mk} - \delta^i_m y_k) + \mu_k(y^i g_{hm} - \delta^i_h y_m) = 0.$

Thus, we conclude

Theorem 2.5. *In $G(\mathcal{B}R) - R-$ affinely connected space, if the directional derivative of covariant vectors field vanish, we have the identity (2.18)*.

Transvecting (22) [5]

$$\mathcal{B}_m H^i_{jkh} + H^r_{kh}G^i_{mjr} - H^i_{rh}G^r_{mjk} - H^i_{kr}G^r_{mjh} = (\dot\partial_j\lambda_m)H^i_{kh} + \lambda_m H^i_{jkh} + (\dot\partial_j\mu_m)(y^i g_{kh} - \delta^i_k y_h) + \mu_m(\delta^i_j g_{kh} - \delta^i_k g_{jh}) + 2y^i\mu_m C_{jkh}$$

by y^j, using (1.9), (1.11), (1.4), (1.1) and (1.6), we get

(2.19) $\mathcal{B}_m H^i_{kh} = \lambda_m H^i_{kh} + \mu_m(y^i g_{kh} - \delta^i_k y_h)$.

Thus, we conclude

Theorem 2.6. *In $G(\mathcal{B}R) - R-$ affinely connected space, if the directional derivative of covariant vectors field vanish, then the h(v) torsion tensor H^i_{kh} is $G\mathcal{B} - R$* .

Transvecting (2.19) by y^k, using (1.9), (1.10), (1.1) and (1.4), we get

(2.20) $\mathcal{B}_m H^i_h = \lambda_m H^i_h$.

Contracting the indices i and h in (2.19), using (1.12), (1.1) and (1.3), we get

(2.21) $\mathcal{B}_m H_k = \lambda_m H_k$.

Contracting the indices i and h in (2.20) and using (1.13), we get

(2.22) $H = \lambda_m H$.

Thus, we conclude

Theorem 2.7. *In $G(\mathcal{B}R) - R-$ affinely connected space, if $\dot\partial_j\lambda_m = 0$, $\dot\partial_j\mu_m = 0$, then the deviation tensor H^i_h, the curvature vector H_k and the* *curvature scalar H are behave as recurrent tensors* .

Transvecting (2.19) by g_{ip} , using (2.3), (1.14), (1.1) and (1.5), we get

(2.23) $\mathcal{B}_m H_{kp.h} = \lambda_m H_{kp.h} + \mu_m (y_p g_{kh} -$ $yhgkp$.

Thus, we conclude

Theorem 2.8. *In $G(\mathcal{B}R) - R -$ affinely connected space, Berwald covariant derivative of first order for the associative torsion tensor $H_{kp.h}$ is given by (2.23). [provided the directional derivative of covariant vectors field vanish].*

3. The Projection On Indicatrix in $G(\mathcal{B}R) - R -$ *Affinely Connected Space*

M. A. A. Ali [12] discussed the projection on indicatrixin$K^h -$birecurrent affinely connected space. A.M.A. Hanballa [2] discussed the projection on indicatrixin the $P^h -$ birecurrent, $P^h -$generalized birecurrent and $P^h -$special generalized birecurrent with respect to Cartan's connection.

Our aim is to discuss the projection on indicatrix for the $H -$ Ricci tensor H_{kh} which behaves as recurrent in $(\mathcal{B}R) - R -$ *affinely connected space* .

The projection of any tensor T^i_j on the indicatrix is given by

(3.1) a) $p . T^i_j := T^a_b \; h^i_a \, h^b_j$,

where

b) $h^i_j := \delta^i_j - l^i l_j$.

The projection of the vector y^i, the unit vectorl^i and the metric tensorg_{ij} on the indicatrix are given by

(3.2) a) $p. y^i = 0$,

b) $p. l^i = 0$

and

c) $p.\ g_{ij} = h_{ij}$,

where

d)$h_{ij} := g_{ij} - l_i l_j$.

Let us consider $G(\mathcal{B}R) - R -$ affinely connected space.

Since in $G(\mathcal{B}R) - R -$ affinely connected space, the $H -$ Ricci tensor H_{kh} behaves as recurrent, i.e. satisfies the condition (2.15).

In view of (3.1a), the projection of the $H -$ Ricci tensor H_{kh} is given by

(3.3) $p. H_{kh} = H_{ab} h^a_k h^b_h$.

Taking the covariant derivative for the condition (3.3) with respect to x^m, in the sense of Berwald , we get

(3.4) $\mathcal{B}_m (p. H_{kh}) = \mathcal{B}_m (H_{ab} h^a_k h^b_h)$.

Using the condition (2.15) and the fact that h^a_b is covariant constant in (3.4), we get

(3.5) $\mathcal{B}_m (p. H_{kh}) = \lambda_m H_{ab} h^a_k h^b_h$.

Using (3.3) in (3.5), we get

(3.6) $\mathcal{B}_m (p. H_{kh}) = \lambda_m (p. H_{kh})$.

Thus, we conclude

Theorem 3.1. *In $G(\mathcal{B}R) - R -$ affinely connected space, the projection of the H –Ricci tensor H_{kh} on indicatrix is $recurrent$.*

Now, let us consider a Finsler space F_n for which the projection of the $H -$ Ricci tensor H_{kh}on indicatrix behaves as recurrent with respect to Berwald's connection i.e. characterized by the condition (3.6).

Using (3.1a) in (3.6), we get

(3.7) $\mathcal{B}_m (H_{ab} h^a_k h^b_h) = \lambda_m (H_{ab} h^a_k h^b_h)$.

Using (3.1b) in (3.7), we get

(3.8) $\mathcal{B}_m \{H_{kh} (\delta^a_k \delta^b_h - \delta^a_k l^b l_h -$ $la l k\delta hb + la l k lb l h = \lambda m$ $Hkh\delta ka\delta hb - \delta ka lb l h - la l k\delta hb + la l k lb l h$.

Using (1.8) in (3.8), we get

(3.9) $\mathcal{B}_m \Big(H_{kh} - H_{kb} \frac{1}{F} y^b l_h - H_{ah} \frac{1}{F} y^a l_k +$ $Hab 1F2 ya yb l k l h$

$= \lambda_m \Big(H_{kh} - H_{kb} \frac{1}{F} y^b l_h - H_{ah} \frac{1}{F} y^a l_k +$ $Hab 1F2 ya yb l k l h$.

Now, if$H_{kb} y^b = 0 = H_{ah} y^a$, then the equation (3.9) becomes

(3.10) $\mathcal{B}_m H_{kh} = \lambda_m H_{kh}$.

Thus, we conclude

Theorem 3.2. *if the projection of the H –Ricci tensor H_{kh}on indicatrix is recurrent, then the H –Ricci tensor H_{kh} itself recurrent, provided$H_{kb} y^b = 0 = H_{ah} y^a$*

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