Notes on Generalized Closed Sets Associated With Digraphs

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Abstract: The aim of this paper is to generate a natural topological structure using digraphs. We give some new generalization of some definitions in digraph to the some known definitions in topology which are $G^d$ -regular open, $G^d$ -open, $G^d$ -semi open, $G^d$ -pre open and $G^d$ -β open. Also we introduce some generalized closed sets in topological spaces associated to the digraph. In particular, some relations between generalized closed sets in topological spaces associated to the digraph are characterized.

Keywords and Phrases : Digraphs, Open sets, Generalized closed sets, Topological spaces.

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1 INTRODUCTION:

The relation between topology and graph theory is undergone many investigations. Many papers are published about this concept. Some of them are exhibiting the relation between digraphs and topologies.


The aim of this paper is to study the nature and properties of some generalized closed sets in topological spaces on digraphs.

2 OPEN SETS ASSOCIATED TO DIGRAPHS

In this section we introduce some weak open sets associated to the digraph $(X, \Gamma)$.

Definition 2.1. A digraph is an ordered pair $(X, \Gamma)$, where X is a set and $\Gamma$ is a binary relation on X. A topology may be determined on a set X by suitable defining subsets of X to be open with respect to the digraph.

A set $A$ of the digraph $(X, \Gamma)$ is $G^d$-open if there exists an edge from $A$ to $A^c$. In other words, a set $A$ of the digraph $(X, \Gamma)$ is $G^d$-open if $p_i \in A$ and $p_j \in A^c$ imply that $p_ip_j \in \Gamma$. A set $A$ of the digraph $(X, \Gamma)$ is $G^d$-closed associated to the digraph (briefly $G^d$-open) if $A^c$ is $G^d$-open. Equivalently, a set $A$ of the digraph $(X, \Gamma)$ is $G^d$-closed if $p_i \in A^c$ and $p_j \in A$ imply that $p_ip_j \in \Gamma$.

Thus each digraph $(X, \Gamma)$ determine a unique topological space $(X, \tau_{G^d})$, where $\tau_{G^d} = \{A: A \subseteq X, (X, \Gamma) is G^d - open\}$

Example 2.2. Consider the following digraph $(X, \Gamma)$ where $X = \{v_1, v_2, v_3, v_4\}$

![Diagrap](image)

Then the topology associated to the above digraph is:

$\tau_{G^d} = \{\emptyset, X, \{v_1\}, \{v_2, v_3\}, \{v_1, v_2, v_3\}\}.$

Definition 2.3. For any set $A \subseteq X$ of the digraph $(X, \Gamma)$, the closure of $A$ with respect to $\tau_{G^d}$ is defined by $cl_{G^d}(A) = \{p_i; p_j accessible to p_i for some p_i \in A. A is G^d - closed if A = cl_{G^d}(A). The interior of $A$ with respect to $\tau_{G^d}$ is defined by $int_{G^d}(A) = X - cl_{G^d}(A^c). A is G^d - open if A = int_{G^d}(A).$
In Example 2.2, \( cl_{\alpha a}(\{v_2\}) = X \), since \( v_2 \) is accessible to \( v_1, v_3, v_4 \). Again
\[ cl_{\alpha a}(\{v_1, v_3\}) = \{v_1, v_3, v_4\} \] Also
\[ int_{\alpha a}(\{v_2, v_4\}) = \{v_2\} \] and
\[ int_{\alpha a}(\{v_1, v_2, v_3\}) = \{v_1, v_2, v_3\} \].

By a similar way of definitions of regular open set (M. Stone, 1937), \( \alpha \)-openset (O. Njastad, 1965), semi-open (N. Levine, 1963), pre-open (A.S. Mashhour et al, 1982) and \( \beta \)-open (M.E. Abd El-Monsef et al, 1983), we introduce the following definitions.

**Definition 2.4.** A subset \( A \) of \( X \) of the digraph \((X, \Gamma)\) and its topological space \((X, \tau_{\alpha a})\) is called
(i) \( G^d \)-regular open set (briefly \( G^d - RO \)) if \( A = int_{\alpha a}(cl_{\alpha a}(A)) \).
(ii) \( G^d \)-semi open set (briefly \( G^d - SO \)) if \( A \subseteq cl_{\alpha a}(int_{\alpha a}(A)) \).
(iii) \( G^d \)-pre open set (briefly \( G^d - PO \)) if \( A \subseteq int_{\alpha a}(cl_{\alpha a}(A)) \).
(iv) \( G^d \)-\( \alpha \) open set (briefly \( G^d - \alpha \)) if \( A \subseteq int_{\alpha a}(cl_{\alpha a}(int_{\alpha a}(A))) \).
(v) \( G^d \)-\( \beta \) open set (briefly \( G^d - \beta O \)) if \( A \subseteq cl_{\alpha a}(int_{\alpha a}(cl_{\alpha a}(A))) \).

The complement of an \( G^d - RO \) (resp. \( G^d - SO, G^d - PO, G^d - \alpha O \) and \( G^d - \beta O \)) is called \( G^d - R \) closed (briefly \( G^d - RC \)) (resp. \( G^d - SC, G^d - PO, G^d - \alpha C \) and \( G^d - \beta C \)).

The family of all \( G^d - RO \) (resp. \( G^d - SO, G^d - PO, G^d - \alpha O \) and \( G^d - \beta O \)) of \((X, \tau_{\alpha a})\) is denoted by \( G^d - RO(X) \) (resp. \( G^d - SO(X), G^d - PO(X), G^d - \alpha O(X) \) and \( G^d - \beta O(X) \)).

\( G^d \)-regular closure of \( A \) (resp. \( G^d \)-semi closure, \( G^d \)-pre closure, \( G^d \)-\( \alpha \) closure and \( G^d \)-\( \beta \) closure) with respect to \( \tau_{\alpha a} \) is denoted by \( cl_{\tau_{\alpha a}}(A) \) (resp. \( cl_{\alpha a}(A), cl_{\alpha a}(A), cl_{\alpha a}(A), cl_{\beta a}(A) \)).

**Example 2.5.** Consider the following digraph \((X, \Gamma)\), where \( X = \{v_1, v_2, v_3, v_4\} \).

\[ \tau_{\alpha a} = \{\varphi, X, \{v_1, v_2\}\} \]
\[ G^d - RO(X) = \{\varphi, X\} \text{ Gd-R(X)} = \{\varphi, X\} \]
\[ G^d - RO(X) = \{\varphi, X, \{v_1, v_3\}, \{v_1, v_4\}, \{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_3, v_4\}/\{v_2, v_3, v_4\}\} \]
\[ G^d - SO(X) = \{\varphi, X, \{v_1, v_3\}, \{v_1, v_4\}, \{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_3, v_4\}/\{v_2, v_3, v_4\}\} \]
\[ G^d - PO(X) = \{\varphi, X, \{v_1, v_3\}, \{v_1, v_4\}, \{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_3, v_4\}/\{v_2, v_3, v_4\}\} \]
\[ G^d - \beta O(X) = \{\varphi, X, \{v_1, v_3\}, \{v_1, v_4\}, \{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_3, v_4\}/\{v_2, v_3, v_4\}\} \]

**Proposition 2.6.** Let \((X, \Gamma)\) be a digraph and \((X, \tau_{\alpha a})\) be the topological space then the following statements are true.
(a) \( G^d - RO(X) \subseteq G^d - O(X) \subseteq G^d - SO(X) \subseteq G^d - \beta O(X) \)
(b) \( G^d - RO(X) \subseteq G^d - O(X) \subseteq G^d - PO(X) \subseteq G^d - \beta O(X) \)
(c) \( G^d - RC(X) \subseteq G^d - C(X) \subseteq G^d - SC(X) \subseteq G^d - \beta C(X) \)
(d) \( G^d - RC(X) \subseteq G^d - C(X) \subseteq G^d - PC(X) \subseteq G^d - \beta C(X) \)
Proof. (a). (i) By definition of $G^d$-regular open, we have $G^d - \text{RO}(X) \subseteq G^d - \text{RO}(X)$

(ii) Let $A$ be any $G^d$-open set in $(X, \tau_G)$, implies $A = \text{int}_{G^d}(A)$, but $\text{int}_{G^d}(A) \subseteq \text{cl}_{G^d} \left( \text{int}_{G^d}(A) \right)$, so $A \subseteq \text{cl}_{G^d} \left( \text{int}_{G^d}(A) \right)$ and $\text{int}_{G^d}(A) \subseteq \text{cl}_{G^d} \left( \text{int}_{G^d}(A) \right)$. Hence $A \subseteq \text{int}_{G^d} \left( \text{cl}_{G^d} \left( \text{int}_{G^d}(A) \right) \right)$ and $A$ is $G^d$-open in $(X, \tau_G)$.

(iii) Let $A$ be any $G^d - \alpha$ open set in $(X, \tau_G)$. $A \subseteq \text{int}_{G^d} \left( \text{cl}_{G^d} \left( \text{int}_{G^d}(A) \right) \right)$. Since $\text{int}_{G^d}(A) \subseteq A$, this implies $\text{int}_{G^d} \left( \text{cl}_{G^d} \left( \text{int}_{G^d}(A) \right) \right) \subseteq \text{cl}_{G^d} \left( \text{int}_{G^d}(A) \right)$. Hence $A \subseteq \text{cl}_{G^d} \left( \text{int}_{G^d}(A) \right)$.

(iv) Let $A$ be $G^d$-semi open in $(X, \tau_G)$, then $A \subseteq \text{cl}_{G^d} \left( \text{int}_{G^d}(A) \right)$. Since $A \subseteq \text{cl}_{G^d}(A)$, this implies $\text{cl}_{G^d} \left( \text{int}_{G^d}(A) \right) \subseteq \text{cl}_{G^d} \left( \text{int}_{G^d}(A) \right)$. Therefore $A$ is $G^d$-open.

(b). (i) Let $A$ be any $G^d$-open in $(X, \tau_G)$, then $A \subseteq \text{int}_{G^d} \left( \text{cl}_{G^d} \left( \text{int}_{G^d}(A) \right) \right)$. Since $\text{int}_{G^d}(A) \subseteq A$, this implies $\text{int}_{G^d} \left( \text{cl}_{G^d} \left( \text{int}_{G^d}(A) \right) \right) \subseteq \text{cl}_{G^d} \left( \text{int}_{G^d}(A) \right)$. Hence $A \subseteq \text{cl}_{G^d} \left( \text{int}_{G^d}(A) \right)$. Therefore $A$ is $G^d$-pre open.

(ii) Let $A$ be $G^d$-pre open in $(X, \tau_G)$, then $A \subseteq \text{int}_{G^d} \left( \text{cl}_{G^d}(A) \right)$. But $\text{cl}_{G^d}(A) \subseteq \text{cl}_{G^d} \left( \text{int}_{G^d}(A) \right)$, implies $A \subseteq \text{cl}_{G^d} \left( \text{cl}_{G^d}(A) \right)$. Therefore $A$ is $G^d$-open.

(c) and (d) Clear.

Remark 2.7. The converse of the above proposition need not be true as the following example illustrates.

Example 2.8. In example 2.5, $v_1, v_2$ is $G^d$-open but not $G^d - R$ open. Also $v_1, v_2, v_4$ is $G^d - \alpha$ open but not $G^d$-open. Again $\{v_2, v_3\}$ is $G^d$-pre open but not $G^d - \alpha$ open. Also $\{v_2, v_4\}$ is $G^d - \beta$ open but not $G^d$-semi open.

Consider the following digraph $(X, \Gamma)$, where $X = \{v_1, v_2, v_3, v_4\}$.

In this section we introduce some generalized closed sets associated to the digraph $(X, \Gamma)$.

By a similar way of definitions of generalized closed set [7], semi-generalized closed set [4], generalized-semi closed set [2], generalized - $\alpha$ closed set [8] and $\alpha$-generalized closed set, we introduce the following definitions.

Definition 3.1. A subset of $X$ of the digraph $(X, \Gamma)$ and its topological Space $(X, \tau_G)$ is called...
(i) generalized closed set associated to the digraph (briefly $G^d - g$ closed) if $cl_{g}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $G^d$-open in $(X, \tau_{g^d})$

(ii) semi-generalized closed set associated to the digraph (briefly $G^d - sg$ closed) if $cl_{sg}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $G^d$-open in $(X, \tau_{g^d})$

(iii) generalized-semi closed set associated to the digraph (briefly $G^d - gs$ closed) if $cl_{gs}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $G^d$-open in $(X, \tau_{g^d})$

(iv) generalized-alclosed set associated to the digraph (briefly $G^d - g\alpha$) if $cl_{a\alpha}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $G^d, \alpha$ open in $(X, \tau_{g^d})$

(v) $\alpha$-generalized closed set associated to the digraph (briefly $G^d - \alpha g$ Gd) if $cl_{ag}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $G^d$-open in $(X, \tau_{g^d})$

The complement of $G^d - g$ closed (resp. $G^d - sg$ closed, $G^d - gs$ closed, $G^d - g\alpha$ closed and $G^d - \alpha g$ closed) set is called $G^d - g$ open (resp. $G^d - sg$ open, $G^d - gs$ open, $G^d - g\alpha$ open and $G^d\alpha g$ open).

**Example 3.2.** Consider the following digraph $(X, \Gamma)$, where $X = \{v_1, v_2, v_3\}$

\[
\tau_{g^d} = \{\phi, X, \{v_1\}, \{v_1, v_2\}\}
\]

**Theorem 3.3.** Every $G^d$ closed set is $G^d - g$ closed in $(X, \tau_{g^d})$.

Proof. Let A be any $G^d$ closed set and U be any $G^d$-open set containing A. Since A is $G^d$ closed, $cl_{g^d}(A) = A$ for every subset $A$ of $X$. Therefore $cl_{g^d}(A) \subseteq U$ and hence A is $G^d - g$ closed.

**Remark 3.4.** The converse of the above theorem need not be true as the following example illustrates.

**Example 3.5.** Consider the digraph in example 3.2, \{v_1, v_3\} is $G^d - g$ closed but not $G^d$-closed.

**Theorem 3.6.** Every $G^d - g$ closed set is $G^d - \alpha g$ closed in $(X, \tau_{g^d})$.

Proof. It is true that $cl_{\alpha g^d}(A) \subseteq cl_{g^d}(A)$ for every subset $A$ of $(X, \tau_{g^d})$.

In general the converse of the above theorem is not true as the following example illustrates.

**Example 3.7.** In example 3.2, we have $v_2$ is $G^d - \alpha g$ closed but not $G^d - g$ closed.

**Theorem 3.8.** Every $G^d - \alpha$ closed set is $G^d - g\alpha$ closed set in $(X, \tau_{g^d})$.

Proof. Let A be any $G^d - \alpha$ closed set and U be any $G^d - \alpha$ open set containing A. Since A is $G^d - \alpha$ closed, $cl_{\alpha g^d}(A) = A$ for every subset $A$ of $X$. Therefore $cl_{\alpha g^d}(A) \subseteq U$ and hence A is $G^d - g\alpha$ closed.

**Theorem 3.9.** Every $G^d$-semi closed set is $G^d - sg$ closed set in $(X, \tau_{g^d})$.

Proof. Let A be an $G^d$-semi closed set and U be any $G^d$-semi open set containing A. Since A is $G^d$-semi closed, $cl_{sg}(A) = A$ for every subset $A$ of $X$. Hence $cl_{sg}(A) \subseteq U$ and therefore A is $G^d - sg$ closed.

**Remark 3.10.** The converse of Theorem 3.8 and Theorem 3.9 are not true in general as the following example illustrates.

**Example 3.11.** Consider the following digraph (X, $\Gamma$) where $X = \{v_1, v_2, v_3\}$
Here \( \{v_1, v_3\} \) is \( G^d - g\alpha \) closed but not \( G^d - \alpha \) closed. Also \( \{v_2, v_3\} \) is \( G^d - sg \) closed but not \( G^d - gs \) closed.

**Theorem 3.12.** Every \( G^d - \alpha g \) closed set is \( G^d - gs \) closed in \( (X, \tau_{G^d}) \).

**Proof.** It is true that every \( G^d \)-open is \( G^d - \alpha g \) open and also \( cl_{\alpha g \alpha}(A) \subseteq cl_{\alpha g \alpha}(A) \) in \( (X, \tau_{G^d}) \).

**Remark 3.13.** The converse of the above Theorem is not true in general as the following example illustrates.

**Example 3.14.** Consider the following digraph \( (X, I) \) where \( X = \{v_1, v_2, v_3\} \).

\[
\tau_{G^d} = \{\varphi, X, \{v_1, v_2, \{v_3\}, \{v_1, v_3\}, \{v_2, v_3\}\}
\]
\[
G^d - \alpha g \text{ closed } = \{\varphi, X, \{v_3\}, \{v_1, v_3\}, \{v_2, v_3\}\}
\]
\[
G^d - gs \text{ closed } = \{\varphi, X, \{v_1\}, \{v_3\}, \{v_1, v_3\}, \{v_2, v_3\}\}
\]

Then \( \{v_2\} \) is \( G^d - gs \) closed but not \( G^d - \alpha g \) closed.

**Theorem 3.15.** Every \( G^d - ga \) closed set is \( G^d - \alpha g \) closed in \( (X, \tau_{G^d}) \).

**Proof.** It is true that every \( G^d \)-open is \( G^d - \alpha g \) open in \( (X, \tau_{G^d}) \).

**Theorem 3.16.** Every \( G^d - sg \) closed set is \( G^d - gs \) closed in \( (X, \tau_{G^d}) \).

**Proof.** It is true that every \( G^d \)-open is \( G^d - \alpha g \) closed in \( (X, \tau_{G^d}) \).

**Remark 3.17.** The converse of Theorem 3.15 and Theorem 3.16 need not be true as shown in the following example.

**Example 3.18.** In Example 3.2, we have \( \{v_1, v_3\} \) is \( G^d - \alpha g \) closed but not \( G^d - ga \) closed. Again \( \{v_2, v_3\} \) is \( G^d - gs \) closed but not \( G^d - g\alpha \) closed.

**References**


